

Fluctuations in the Uniform Shear Flow state of a granular gas

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Abstract

We study the fluctuations of the total internal energy of a granular gas under stationary uniform shear flow by means of kinetic theory methods. We find that these fluctuations are coupled to the fluctuations of the different components of the total pressure tensor. Explicit expressions for all the possible cross and auto correlations of the fluctuations at one and two times are obtained in the two dimensional case. The theoretical predictions are compared with Molecular Dynamics simulation and a good agreement is found for the range of inelasticity considered.

I. INTRODUCTION

A granular system can be defined as an ensemble of macroscopic particles (grains) that collide inelastically, i.e. kinetic energy is dissipated in collisions. When the dynamics of the grains can be partitioned into sequences of two-body collisions, the system is referred to a granular gas and there is support both from experiments and computer simulations of the reliability of a kinetic theory description [1–5]. One of the most used models to study granular gases is the Inelastic Hard Sphere (IHS) model, whose dynamics is given in terms of free streaming followed by instantaneous inelastic collisions. For this model, all the kinetic theory machinery can be applied [6]. In particular, in the low density limit the dynamics of the one-particle distribution function is given by the inelastic Boltzmann equation [7, 8] and the correlation functions obey a closed set of equations [9].

Macroscopically, it is known that, in many cases, the dynamics of a granular system is reminiscent of that of a fluid. For dilute systems, hydrodynamic equations can be derived applying the Chapman-Enskog expansion [10] or linear response methods [11–13], obtaining explicit expressions for the transport coefficients. In all these studies, there is a particular state which plays a specially important role; the Homogeneous Cooling State (HCS). This is a homogeneous state in which all the time dependence in the one-particle distribution function goes through the granular temperature (defined as the second velocity moment of the velocity distribution). Due to the inelasticity of collisions, the temperature decays monotonically in time [14]. It is known that, for a wide class of initial conditions, the HCS is reached in the long-time limit for isolated granular gases. This fact makes that this state play, for granular gases, a similar role to the equilibrium state in the context of molecular, elastic fluids. In fact, the zeroth order in the gradients distribution in the Chapman-Enskog

expansion is a “local” HCS [10].

Despite this analogy with normal fluids, there are also important differences. Due to the macroscopic character of the grains, a granular system contains typically much less particles than a normal fluid. This fact makes that the fluctuations of the macroscopic fields be of special relevance not only theoretically, but also from a practical point of view. The fluctuations of the total energy have been studied in the HCS, and explicit expressions for its variance and two-time correlation function have been obtained [9]. With some generality, Langevin-like equations for the fluctuating hydrodynamic fields have been derived to Navier-Stokes order [15, 16], finding that there are not Fluctuation-Dissipation theorems of the second kind, i.e. the amplitude of the noises are not related to the transport coefficients. On the other hand, the two-time correlation functions do decay as a macroscopic perturbation so that Fluctuation-Dissipation theorems of first kind hold [17].

The study of the fluctuations in the HCS is of special relevance, because it serves as a starting point for the generalization to other states. Making an analogy with normal fluids, the equations for the fluctuating fields can be written in an intuitive manner for states that are close to the HCS. The deterministic part of the equations is the linearization of the macroscopic equations around the particular state considered. The noises can be assumed to have the same stochastic properties that in the HCS but replacing the total fields by the local actual ones. As said, this is expected to be valid if the state is not far from the HCS, which means small gradients. In fact, this idea was applied in [18] to calculate the total internal energy fluctuations in the stationary Uniform Shear Flow (USF) state. This state is characterized by a uniform density, a constant and uniform temperature, and a flow velocity with a linear profile and, due to its simplicity, it has been extensively studied [19–24]. The theoretical predictions of [18] were expected to hold only for small gradients, that for the USF means small inelasticity due to the coupling between gradients and inelasticity, which is a characteristic feature of stationary states of granular systems. The objective of this work is the study of the fluctuations of the total internal energy in the stationary USF state using kinetic theory tools. This will let us analyze the problem in general (without any limitation to small inelasticity) and, in particular, the differences with the “local” HCS results of [18]. It will be shown that the structure of these fluctuations is more complex than expected, since they are coupled to the fluctuations of the several components of the total pressure tensor. In the end, a systematic and controlled expansion in the degree of inelasticity will

be done, in order to be able to get explicit results.

The plan of the paper is as follows. In the next section, the IHS model is described in some detail, and the evolution equations for the relevant distributions are summarized. These equations are applied to the stationary USF state in section III, where the specific case of correlations of global quantities is considered. In section IV we study the fluctuations of the total internal energy and it is shown that they are coupled to the components of the total pressure tensor as mentioned above. The complete study of all the fluctuations is carried out in section V. The analytical predictions are compared to Molecular Dynamics simulation results in section VI, finding, in general, a good agreement. The final section contains some general conclusions and comments.

II. KINETIC EQUATIONS FOR THE MODEL

The system we consider is a dilute gas of N smooth inelastic hard spheres ($d = 3$) or disks ($d = 2$) of mass m and diameter σ . Let $X_i(t) \equiv \{\mathbf{R}_i(t), \mathbf{V}_i(t)\}$ denote the position and velocity of particle i at time t . The dynamical state of the system, $\Gamma(t) \equiv \{X_1(t), \dots, X_N(t)\}$, is generated by free streaming followed by instantaneous inelastic collisions characterized by a coefficient of normal restitution, α , independent of the relative velocity. If at time t there is a binary encounter between particles i and j , with velocities $\mathbf{V}_i(t)$ and $\mathbf{V}_j(t)$ respectively, the postcollisional velocities $\mathbf{V}'_i(t)$ and $\mathbf{V}'_j(t)$ are

$$\begin{aligned}\mathbf{V}'_i &= \mathbf{V}_i - \frac{1+\alpha}{2}(\hat{\boldsymbol{\sigma}} \cdot \mathbf{V}_{ij})\hat{\boldsymbol{\sigma}}, \\ \mathbf{V}'_j &= \mathbf{V}_j + \frac{1+\alpha}{2}(\hat{\boldsymbol{\sigma}} \cdot \mathbf{V}_{ij})\hat{\boldsymbol{\sigma}},\end{aligned}\tag{1}$$

where $\mathbf{V}_{ij} \equiv \mathbf{V}_i - \mathbf{V}_j$ is the relative velocity and $\hat{\boldsymbol{\sigma}}$ is the unit vector pointing from the center of particle j to the center of particle i at contact.

Microscopic densities in phase space, $F_s(x_1, \dots, x_s, t)$, are defined by

$$F_1(x_1, t) = \sum_{i=1}^N \delta[x_1 - X_i(t)],\tag{2}$$

$$F_2(x_1, x_2, t) = \sum_{i=1}^N \sum_{j \neq i}^N \delta[x_1 - X_i(t)] \delta[x_2 - X_j(t)],\tag{3}$$

etc, where we have introduced the field variables, $x_i \equiv \{\mathbf{r}_i, \mathbf{v}_i\}$. The averages of the microscopic densities over the probability distribution function, $\rho(\Gamma, 0)$, characterizing the initial

state are the usual one-time reduced distribution functions

$$f_s(x_1, \dots, x_s, t) \equiv \langle F_s(x_1, \dots, x_s, t) \rangle, \quad (4)$$

where we have introduced the notation

$$\langle G \rangle \equiv \int d\Gamma G(\Gamma) \rho(\Gamma, 0). \quad (5)$$

Two-time reduced distribution functions can also be defined in terms of the microscopic densities as

$$f_{r,s}(x_1, \dots, x_r, t; x_1, \dots, x_s, t') \equiv \langle F_r(x_1, \dots, x_r, t) F_s(x_1, \dots, x_s, t') \rangle, \quad (6)$$

where it will be assumed that $t > t' > 0$ for concreteness. Evolution equations for the reduced distributions can be derived from first principles [8, 9], in the same way as in the elastic case [25]. The one-time reduced distribution functions obey the generalization for inelastic collisions of the Bogoliubov, Born, Green, Kirkwood and Yvon hierarchy, but its application in general is limited due to the fact that the equations are not closed. The same occurs for the two-time reduced distribution functions.

It is convenient to introduce correlation functions through the usual cluster expansion. From the one-time reduced distributions, one-time correlations, $g_s(x_1, \dots, x_s, t)$, are defined by

$$f_2(x_1, x_2, t) \equiv f_1(x_1, t) f_1(x_2, t) + g_2(x_1, x_2, t), \quad (7)$$

$$\begin{aligned} f_3(x_1, x_2, x_3, t) &\equiv f_1(x_1, t) f_1(x_2, t) f_1(x_3, t) + f_1(x_1, t) g_2(x_2, x_3, t) \\ &+ f_1(x_2, t) g_2(x_1, x_3, t) + f_1(x_3, t) g_2(x_1, x_2, t) + g_3(x_1, x_2, x_3, t), \end{aligned} \quad (8)$$

etc. Similarly, two-time correlations functions, $h_{r,s}(x_1, \dots, x_r, t; x_1, \dots, x_s, t')$, can be defined. In particular, $h_{1,1}$ and $h_{2,1}$ are introduced through

$$f_{1,1}(x_1, t; x'_1, t') = f_1(x_1, t) f_1(x'_1, t') + h_{1,1}(x_1, t; x'_1, t'), \quad (9)$$

$$\begin{aligned} f_{2,1}(x_1, x_2, t; x'_1, t') &= f_1(x_1, t) f_1(x_2, t) f_1(x'_1, t') + g_2(x_1, x_2, t) f_1(x'_1, t') \\ &+ h_{1,1}(x_1, t; x'_1, t') f_1(x_2, t) + h_{1,1}(x_2, t; x'_1, t') f_1(x_1, t) + h_{2,1}(x_1, x_2, t; x'_1, t'). \end{aligned} \quad (10)$$

In the low density limit and for distances much longer than the diameter of the particles, a closed set of equations for f_1 , g_2 and $h_{1,1}$ is obtained [9, 25].

The one particle distribution function satisfies the inelastic Boltzmann equation [7, 8]

$$\left[\frac{\partial}{\partial t} + L^{(0)}(x_1) \right] f_1(x_1, t) = J[x_1, t|f_1], \quad (11)$$

where we have introduced the free-streaming operator

$$L^{(0)}(x_1) = \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{r}_1}. \quad (12)$$

The collisional term reads

$$J[x_1, t|f_1] = \int dx_2 \delta(\mathbf{r}_{12}) \bar{T}_0(\mathbf{v}_1, \mathbf{v}_2) f_1(x_1, t) f_1(x_2, t), \quad (13)$$

with the binary collision operator, T_0 , given by

$$\bar{T}_0(\mathbf{v}_1, \mathbf{v}_2) = \sigma^{d-1} \int d\hat{\boldsymbol{\sigma}} \Theta(\mathbf{v}_{12} \cdot \hat{\boldsymbol{\sigma}}) (\mathbf{v}_{12} \cdot \hat{\boldsymbol{\sigma}}) [\alpha^{-2} b_{\boldsymbol{\sigma}}^{-1}(1, 2) - 1]. \quad (14)$$

Here $\boldsymbol{\sigma} = \sigma \hat{\boldsymbol{\sigma}}$, $d\hat{\boldsymbol{\sigma}}$ is the solid angle element for $\hat{\boldsymbol{\sigma}}$, $\mathbf{v}_{12} \equiv \mathbf{v}_1 - \mathbf{v}_2$, Θ is the Heaviside step function and the operator $b_{\boldsymbol{\sigma}}^{-1}(1, 2)$ replaces all the velocities \mathbf{v}_1 and \mathbf{v}_2 appearing to its right by the precollisional values \mathbf{v}_1^* and \mathbf{v}_2^* ,

$$\begin{aligned} \mathbf{v}_1^* &\equiv b_{\boldsymbol{\sigma}}^{-1}(1, 2) \mathbf{v}_1 = \mathbf{v}_1 - \frac{1 + \alpha}{2\alpha} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{v}_{12}) \hat{\boldsymbol{\sigma}}, \\ \mathbf{v}_2^* &\equiv b_{\boldsymbol{\sigma}}^{-1}(1, 2) \mathbf{v}_2 = \mathbf{v}_2 + \frac{1 + \alpha}{2\alpha} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{v}_{12}) \hat{\boldsymbol{\sigma}}. \end{aligned} \quad (15)$$

The equation for the one-time correlation function in the low density limit is

$$\begin{aligned} \left[\frac{\partial}{\partial t} + L^{(0)}(x_1) + L^{(0)}(x_2) - K[x_1, t|f_1] - K[x_2, t|f_1] \right] g_2(x_1, x_2, t) \\ = \delta(\mathbf{r}_{12}) \bar{T}_0(\mathbf{v}_1, \mathbf{v}_2) f_1(x_1, t) f_1(x_2, t), \end{aligned} \quad (16)$$

where we have introduced the linear operator

$$K[x_i, t|f_1] \equiv \int dx_3 \delta(\mathbf{r}_{i3}) \bar{T}_0(\mathbf{v}_i, \mathbf{v}_3) (1 + P_{i3}) f_1(x_3, t). \quad (17)$$

The operator P_{ij} interchanges the labels of particle i and j in the quantities to its right. Basically, Eq. (16) shows that velocity correlations between particles with velocities \mathbf{v}_1 and \mathbf{v}_2 are generated by uncorrelated collisions implying particles with velocities \mathbf{v}_1 and \mathbf{v}_2 through the right hand side of Eq. (16), and correlated collisions implying two particles with velocities \mathbf{v}_1 or \mathbf{v}_2 and a third particle with velocity \mathbf{v}_3 through the linear operator K .

Finally, let us consider the two-time correlations. In the same limit, the following equation for the first two-time correlation function is obtained

$$\left[\frac{\partial}{\partial t} + L^{(0)}(x_1) - K[x_1, t|f_1] \right] h_{1,1}(x_1, t; x'_1, t') = 0. \quad (18)$$

This equation has to be solved with the initial condition

$$h_{1,1}(x_1, t'; x'_1, t') = f_1(x_1, t')\delta(x_1 - x'_1) + g_2(x_1, x'_1, t'), \quad (19)$$

that follows directly from the definitions in Eqs. (6) and (9). Let us remark that if we consider states with a one-particle distribution function, \tilde{f}_1 , very closed to a given reference distribution, f_1 , the difference of both distributions, $\delta f_1 \equiv \tilde{f}_1 - f_1$, fulfills to linear order

$$\left[\frac{\partial}{\partial t} + L^{(0)}(x_1) - K[x_1, t|f_1] \right] \delta f_1(x_1, t) = 0, \quad (20)$$

where K is given by Eq. (17). The structure of these equations is important because it shows that the two-time correlations in a given state decays in the same way that a linear perturbation of the one-particle distribution function around this state. This is so because the linear operator governing the dynamics is, in both cases, the operator K given by Eq. (17). Of course, although the initial condition of Eq. (20) is free (with the only restriction to have $|\delta f_1(x_1, t')| \ll f_1(x_1, t')$), the initial condition for Eq. (18) is given by Eq. (19).

III. THE STATIONARY UNIFORM SHEAR FLOW STATE

In this section we are going to apply the equations of the previous section to a particular state, the stationary USF state. At a macroscopic level, this state is characterized by a uniform number density, n_s , a stationary temperature, T_s , and a constant velocity field with linear profile, $\mathbf{u}_s = ay\hat{\mathbf{e}}_x$, where a is the constant shear rate and $\hat{\mathbf{e}}_x$ is a unit vector in the direction of the x -axis (the subindex s has been introduced to label the state) [20–23]. In this stationary state, the cooling due to collisions is compensated by viscous heating

$$\frac{2a}{dn_s} P_{xy,s} = \zeta_s T_s, \quad (21)$$

where $P_{xy,s}$ is the xy component of the stress tensor and ζ_s is the cooling rate. For a hydrodynamic description, $P_{xy,s}$ and ζ_s have to be expressed in terms of the hydrodynamic fields, n_s , T_s and \mathbf{u}_s , and their gradients, i.e. the shear rate, a [10, 26].

The USF state can be studied by means of kinetic theory. The definitions of the hydrodynamic fields in terms of the one-particle distribution function are the usual in kinetic theory

$$n(\mathbf{r}, t) \equiv \int d\mathbf{v} f_1(x, t), \quad (22)$$

$$n(\mathbf{r}, t) \mathbf{u}(\mathbf{r}, t) \equiv \int d\mathbf{v} \mathbf{v} f_1(x, t), \quad (23)$$

$$\frac{d}{2} n(\mathbf{r}, t) T(\mathbf{r}, t) \equiv \int d\mathbf{v} \frac{m}{2} [\mathbf{v} - \mathbf{u}(\mathbf{r}, t)]^2 f_1(x, t). \quad (24)$$

The expressions of the pressure tensor and cooling rate are [10]

$$P_{ij}(\mathbf{r}, t) = m \int d\mathbf{v} [v_i - u_i(\mathbf{r}, t)][v_j - u_j(\mathbf{r}, t)] f_1(x, t), \quad (25)$$

and

$$\zeta(\mathbf{r}, t) = \frac{(1 - \alpha^2) \pi^{(d-1)/2} m \sigma^{d-1}}{4d\Gamma\left(\frac{d+3}{2}\right) n(\mathbf{r}, t) T(\mathbf{r}, t)} \int d\mathbf{v}_1 \int d\mathbf{v}_2 v_{12}^3 f_1(\mathbf{r}, \mathbf{v}_1, t) f_1(\mathbf{r}, \mathbf{v}_2, t). \quad (26)$$

The Boltzmann equation admits a normal solution of the USF type, i.e. a solution in which all the space dependence goes through the hydrodynamic fields and all their gradients. As the only space-dependent field is linear, the distribution function can be written in a particularly simple form

$$f_{USF}(\mathbf{r}, \mathbf{v}) = f_s[\mathbf{v} - \mathbf{u}_s(\mathbf{r}), n_s, T_s, a]. \quad (27)$$

By substituting this expression into the Boltzmann equation, it follows that

$$aV_{1y} \frac{\partial}{\partial V_{1x}} f_s(\mathbf{V}_1) + \int d\mathbf{V}_2 \bar{T}_0(\mathbf{V}_1, \mathbf{V}_2) f_s(\mathbf{V}_1) f_s(\mathbf{V}_2) = 0, \quad (28)$$

where we have introduced the peculiar velocity $\mathbf{V} \equiv \mathbf{v} - \mathbf{u}_s(\mathbf{r})$, and we have skipped the explicit dependence on the hydrodynamic fields and the shear rate in the distribution function. Note that the form of the distribution given by Eq. (27) implies that the system is homogeneous in the Lagrangian frame of reference. Although the exact solution of Eq. (28) is not known, many approximate solutions are available [20–22, 24]. In this work we will consider the $\epsilon \equiv (1 - \alpha^2)^{1/2}$ expansion of the Jenkins and Richman approximation up to order ϵ^2 . The specific form of the distribution will be given later on.

It is convenient to perform the following change of variables

$$\{\mathbf{r}, \mathbf{v}, t\} \longrightarrow \{\ell(\mathbf{r}, t) = \mathbf{r} - ayt\hat{\mathbf{e}}_x, \mathbf{V}(\mathbf{r}, t) = \mathbf{v} - ay\hat{\mathbf{e}}_x, t\}. \quad (29)$$

Actually, the USF is usually generated in particle simulations using Lees-Edwards boundary conditions [27] and, as stressed in [28], these boundary conditions transform into periodic boundary conditions in the new variables. As the Jacobian of the transformation is one, the function

$$f(\boldsymbol{\ell}, \mathbf{V}, t) = f_1[\mathbf{r}(\boldsymbol{\ell}, t), \mathbf{v}(\boldsymbol{\ell}, \mathbf{V}, t), t], \quad (30)$$

is the actual distribution function in the new variables. Let us consider situations very closed to the USF state, in such a way that the deviations, $\delta f(\boldsymbol{\ell}, \mathbf{V}, t) \equiv f(\boldsymbol{\ell}, \mathbf{V}, t) - f_s(\mathbf{V})$, are assumed to fulfill the condition $|\delta f(\boldsymbol{\ell}, \mathbf{V}, t)| \ll f_s(\mathbf{V})$. To linear order, δf satisfies

$$\frac{\partial}{\partial t} \delta f(\boldsymbol{\ell}, \mathbf{V}_1, t) = H(\boldsymbol{\ell}, \mathbf{V}_1, t) \delta f(\boldsymbol{\ell}, \mathbf{V}_1, t), \quad (31)$$

where

$$H(\boldsymbol{\ell}, \mathbf{V}_1, t) \equiv L(\mathbf{V}_1) - \mathbf{V}_1 \cdot \frac{\partial}{\partial \boldsymbol{\ell}} - a\ell_y \frac{\partial}{\partial \ell_x} + atV_{1y} \frac{\partial}{\partial \ell_x} \quad (32)$$

is an inhomogeneous linear operator with

$$L(\mathbf{V}_1)h(\mathbf{V}_1) \equiv \int d\mathbf{V}_2 \bar{T}_0(\mathbf{V}_1, \mathbf{V}_2)(1 + P_{12})f_s(\mathbf{V}_1)h(\mathbf{V}_2) + aV_{1y} \frac{\partial}{\partial V_{1x}} h(\mathbf{V}_1). \quad (33)$$

Note that, in contrast with the free cooling case [11–13], the inhomogeneous term in Eq. (32) is time dependent.

Consider now the one-time and two-time correlation functions in the USF, g_{2USF} and $h_{1,1USF}$ respectively. In the new variables, the equations for

$$G_s(\boldsymbol{\ell}_1, \mathbf{V}_1, \boldsymbol{\ell}_2, \mathbf{V}_2) \equiv g_{2USF}(x_1, x_2), \quad (34)$$

and

$$h_s(\boldsymbol{\ell}_1, \mathbf{V}_1, t; \boldsymbol{\ell}_2, \mathbf{V}_2, t') \equiv h_{1,1USF}(x_1, t; x_2, t'), \quad (35)$$

are

$$[H(\boldsymbol{\ell}_1, \mathbf{V}_1, t) + H(\boldsymbol{\ell}_2, \mathbf{V}_2, t)] G_s(\boldsymbol{\ell}_1, \mathbf{V}_1, \boldsymbol{\ell}_2, \mathbf{V}_2) = -\delta(\boldsymbol{\ell}_{12}) \bar{T}_0(\mathbf{V}_1, \mathbf{V}_2) f_s(\mathbf{V}_1) f_s(\mathbf{V}_2), \quad (36)$$

and

$$\frac{\partial}{\partial t} h_s(\boldsymbol{\ell}_1, \mathbf{V}_1, t; \boldsymbol{\ell}_2, \mathbf{V}_2, t') = H(\boldsymbol{\ell}_1, \mathbf{V}_1, t) h_s(\boldsymbol{\ell}_1, \mathbf{V}_1, t; \boldsymbol{\ell}_2, \mathbf{V}_2, t'), \quad (37)$$

respectively. This last equation has to be solved with the initial condition (see Eq. (19))

$$h_s(\boldsymbol{\ell}_1, \mathbf{V}_1, t'; \boldsymbol{\ell}_2, \mathbf{V}_2, t') = f_s(\mathbf{V}_1) \delta(\boldsymbol{\ell}_{12}) \delta(\mathbf{V}_{12}) + G_s(\boldsymbol{\ell}_1, \mathbf{V}_1, \boldsymbol{\ell}_2, \mathbf{V}_2). \quad (38)$$

Equations (36) and (37) describe two particle correlations at one and two times. Basically, they depend on the one-particle distribution function, which is supposed to be known, and on the linear operator defined by Eq. (33).

A. Global correlations

As the general problem is quite involved, in the following we will focus on a simplified problem: the study of the correlations between global quantities. In order to deal with dimensionless distributions, we introduce the dimensionless velocity

$$\mathbf{c} = \frac{\mathbf{V}}{v_s}, \quad v_s = \sqrt{\frac{2T_s}{m}}, \quad (39)$$

through the thermal velocity in the stationary USF, v_s , and the dimensionless time

$$s = \frac{v_s}{\lambda} t, \quad \lambda = (n_s \sigma^{d-1}), \quad (40)$$

with λ proportional to the mean free path. In terms of these units, we define the dimensionless distributions. The scaled one-particle distribution function in the USF is

$$\chi(\mathbf{c}) \equiv \frac{v_s^d}{n_s} f_s(\mathbf{V}), \quad (41)$$

the integrated deviation of the one-particle distribution function around the USF is

$$\delta\chi(\mathbf{c}, s) \equiv \frac{v_s^d}{n_s} \int d\ell \delta f(\ell, \mathbf{V}, t), \quad (42)$$

the dimensionless marginal one-time correlation function is

$$\phi(\mathbf{c}_1, \mathbf{c}_2) \equiv \frac{v_s^{2d}}{N} \int d\ell_1 \int d\ell_2 G_s(\ell_1, \mathbf{V}_1, \ell_2, \mathbf{V}_2), \quad (43)$$

and the dimensionless marginal two-time correlation function is

$$\psi(\mathbf{c}_1, \mathbf{c}_2, s - s') \equiv \frac{v_s^{2d}}{N} \int d\ell_1 \int d\ell_2 h_s(\ell_1, \mathbf{V}_1, t; \ell_2, \mathbf{V}_2, t'). \quad (44)$$

For homogeneous states in the Lagrangian frame of reference, the evolution equation for $\delta\chi$, obtained by integrating of Eq. (31), reads

$$\frac{\partial}{\partial s} \delta\chi(\mathbf{c}, s) = \Lambda(\mathbf{c}) \delta\chi(\mathbf{c}, s). \quad (45)$$

The operator Λ will be called linearized Boltzmann operator and is the adimensionalization of the linear operator defined in Eq. (33), i.e.,

$$\Lambda(\mathbf{c}_1) h(\mathbf{c}_1) \equiv \int d\mathbf{c}_2 \tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2) (1 + P_{12}) \chi(\mathbf{c}_1) h(\mathbf{c}_2) + \tilde{a}_s c_{1y} \frac{\partial}{\partial c_{1x}} h(\mathbf{c}_1), \quad (46)$$

where \tilde{T}_0 is the dimensionless counterpart of \overline{T}_0

$$\tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2) \equiv \int d\hat{\boldsymbol{\sigma}} \Theta(\mathbf{c}_{12} \cdot \hat{\boldsymbol{\sigma}})(\mathbf{c}_{12} \cdot \hat{\boldsymbol{\sigma}})[\alpha^{-2} b_{\boldsymbol{\sigma}}^{-1}(1, 2) - 1], \quad (47)$$

and

$$\tilde{a}_s = \frac{\lambda a}{v_s}, \quad (48)$$

is the dimensionless shear rate.

The equation for the one-time correlation function is

$$[\Lambda(\mathbf{c}_1) + \Lambda(\mathbf{c}_2)]\phi(\mathbf{c}_1, \mathbf{c}_2) = -\tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2)\chi(\mathbf{c}_1)\chi(\mathbf{c}_2), \quad (49)$$

and the evolution equation for the two-time correlation functions is

$$\frac{\partial}{\partial s}\psi(\mathbf{c}_1, \mathbf{c}_2, s) = \Lambda(\mathbf{c}_1)\psi(\mathbf{c}_1, \mathbf{c}_2, s), \quad (50)$$

to be solved with the initial condition

$$\psi(\mathbf{c}_1, \mathbf{c}_2, 0) = \chi(\mathbf{c}_1)\delta(\mathbf{c}_{12}) + \phi(\mathbf{c}_1, \mathbf{c}_2). \quad (51)$$

It is important to remark the strong analogy between equations (49) and (50), and the equivalent ones in other granular states (as the homogeneous cooling state [9]), or other granular systems in which the particles are accelerated by a stochastic force [29]. The analogy is also evident with other dissipative systems [30], where the linearized Boltzmann collision operator, $\Lambda(\mathbf{c})$, always plays an essential role in the structure of the global correlations.

B. Correlations between global quantities

The correlations between global quantities can be evaluated using the distributions we have introduced above. Consider quantities of the form

$$\mathcal{A}(t) = \sum_{i=1}^N a[\mathbf{V}_i - \mathbf{u}_s(\mathbf{R}_i)] = \int d\mathbf{r} \int d\mathbf{v} a[\mathbf{v} - \mathbf{u}_s(\mathbf{r})] F_1(x, t), \quad (52)$$

where a is supposed to be a homogeneous function of degree β , i.e. $a(k\mathbf{c}) = k^\beta a(\mathbf{c})$. The deviation around the mean in the USF is

$$\delta\mathcal{A}(t) \equiv \mathcal{A}(t) - \langle \mathcal{A}(t) \rangle = \int d\mathbf{r} \int d\mathbf{v} a(\mathbf{V}) \delta F(x, t), \quad (53)$$

where

$$\delta F(x, t) \equiv F_1(x, t) - f_{USF}(x). \quad (54)$$

The correlations between the fluctuations of two different quantities, \mathcal{A}_1 and \mathcal{A}_2 , of the form given in Eq. (52) can be expressed as

$$\langle \delta \mathcal{A}_1(t) \delta \mathcal{A}_2(t') \rangle = \int d\mathbf{r}_1 \int d\mathbf{v}_1 \int d\mathbf{r}_2 \int d\mathbf{v}_2 a_1(\mathbf{V}_1) a_2(\mathbf{V}_2) h_{1,1USF}(x_1, t; x_2, t'). \quad (55)$$

Upon writing this expression, we have used that

$$\langle \delta F(x_1, t) \delta F(x_2, t') \rangle = h_{1,1USF}(x_1, t; x_2, t'). \quad (56)$$

Expressing the integrand of (55) in term of the dimensionless distribution defined in Eq. (44), yields

$$\langle \delta \mathcal{A}_1(t) \delta \mathcal{A}_2(t') \rangle = N v_s^{\beta_1 + \beta_2} \int d\mathbf{c}_1 \int d\mathbf{c}_2 a_1(\mathbf{c}_1) a_2(\mathbf{c}_2) \psi(\mathbf{c}_1, \mathbf{c}_2, s - s'), \quad (57)$$

where β_1 and β_2 are the degree of homogeneity of a_1 and a_2 , respectively.

The expression for the one-time correlations is obtained by performing $s = s'$ in Eq. (57). Taking into account Eq. (51), it follows that

$$\langle \delta \mathcal{A}_1(t) \delta \mathcal{A}_2(t) \rangle = N v_s^{\beta_1 + \beta_2} \left[\int d\mathbf{c} a_1(\mathbf{c}) a_2(\mathbf{c}) \chi(\mathbf{c}) + \int d\mathbf{c}_1 \int d\mathbf{c}_2 a_1(\mathbf{c}_1) a_2(\mathbf{c}_2) \phi(\mathbf{c}_1, \mathbf{c}_2) \right]. \quad (58)$$

IV. FLUCTUATIONS OF THE TOTAL INTERNAL ENERGY

Microscopically, the total internal energy is defined as

$$\mathcal{E}(t) = \sum_{i=1}^N \frac{m}{2} [\mathbf{V}_i - \mathbf{u}_s(\mathbf{R}_i)]^2, \quad (59)$$

so that it is a quantity of the form introduced in the previous section. We identify, $a(\mathbf{V}) \equiv \frac{m}{2} V^2$, that is a homogeneous function of degree two. Using Eq. (58), we get

$$\langle \delta \mathcal{E}^2(t) \rangle = \frac{m^2}{4} N v_s^4 \left[\int d\mathbf{c} c^4 \chi(\mathbf{c}) + \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_1^2 c_2^2 \phi(\mathbf{c}_1, \mathbf{c}_2) \right]. \quad (60)$$

Since the distribution χ is supposed to be known, we only have to evaluate the velocity moment of ϕ that appears in the right hand side of Eq. (60). In order to do that we follow a method based on the analysis of some spectral properties of the linearized Boltzmann collision operator, Λ [9, 29, 30].

A. Spectral properties of Λ

In order to identify some modes of the Λ operator, it is necessary to introduce the time-dependent USF state [31, 32]. As the stationary USF, the time-dependent state is characterized macroscopically by a constant density, n_H , and a time-independent flow velocity, $\mathbf{u}_H(\mathbf{r}) = ay\hat{\mathbf{e}}_x$. The temperature, $T_H(t)$, remains homogeneous, but it is time-dependent. The subindex H distinguishes it from the stationary state labeled by s . By dimensional analysis, if a normal distribution function for this state exists, it has the form

$$f_H(\mathbf{V}, t) = \frac{n_H}{v_H(t)^d} \chi(\mathbf{c}, \tilde{a}), \quad (61)$$

where

$$\mathbf{c} = \frac{\mathbf{V}}{v_H(t)}, \quad v_H(t) = \sqrt{\frac{2T_H(t)}{m}}, \quad \tilde{a} = \frac{\lambda a}{v_H(t)}. \quad (62)$$

We use the same notation for the time-dependent scaled velocity $\mathbf{V}/v_H(t)$ and for \mathbf{V}/v_s but this will not cause any difficulty. In the long time limit, this distribution tends to the stationary one

$$\chi(\mathbf{c}, \tilde{a}) \rightarrow \chi(\mathbf{c}, \tilde{a}_s) \equiv \chi(\mathbf{c}), \quad (63)$$

and also the quantities $v_H(t)$ and \tilde{a} to their stationary values v_s and \tilde{a}_s respectively.

Let us consider the family of states given by Eq. (61) with the restriction of being close to the stationary USF state. These states are characterized by the two parameters

$$\rho \equiv \frac{\delta n}{n_s}, \quad \theta \equiv \frac{\delta T}{T_s}. \quad (64)$$

It is assumed that the deviations

$$\delta n \equiv n_H - n_s, \quad \delta T \equiv T_H - T_s, \quad (65)$$

are small, i.e. $|\delta n| \ll n_s$ and $|\delta T| \ll T_s$. We do not include states with different shear rates, a , because we want all the states to be generated by the same boundary conditions. Performing a similar analysis to the one carried out in reference [33], the following evolution equation for θ

$$\frac{d\theta(s)}{ds} = -\gamma[2\rho + \theta(s)], \quad (66)$$

is obtained in Appendix A. As the total number of particle does not vary, ρ is constant and we can identify the normal mode $[2\rho + \theta(s)]$. The eigenvalue

$$\gamma = \frac{\tilde{\zeta}(\tilde{a}_s)}{2} - \frac{\tilde{a}_s^2}{d} \frac{d\tilde{P}_{xy}}{d\tilde{a}}(\tilde{a}_s) - \frac{\tilde{a}_s}{2} \frac{d\tilde{\zeta}}{d\tilde{a}}(\tilde{a}_s), \quad (67)$$

is expressed in terms of the dimensionless pressure tensor

$$\tilde{P}_{xy}(\tilde{a}) = 2 \int d\mathbf{c} c_x c_y \chi(\mathbf{c}, \tilde{a}), \quad (68)$$

and the dimensionless cooling rate

$$\tilde{\zeta}(\tilde{a}) = \frac{\pi^{(d-1)/2}(1-\alpha^2)}{2d\Gamma(\frac{d+3}{2})} \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{12}^3 \chi(\mathbf{c}_1, \tilde{a}) \chi(\mathbf{c}_2, \tilde{a}), \quad (69)$$

in the time-dependent USF state. Eq. (67) is equivalent to the one derived in [34] and also to the one of [33] for the case $\rho = 0$. An explicit formula for γ as a function of the inelasticity can be written using the expressions of \tilde{P}_{xy} and $\tilde{\zeta}(\tilde{a}_s)$ of the BGK model studied in [32] and neglecting the contribution proportional to $d\tilde{\zeta}/d\tilde{a}$.

B. An unaccurate approximation

Let us rewrite Eq. (66) in a way that suggests the approximation to be analyzed in the following. Define the scalar product

$$\langle h(\mathbf{c}) | g(\mathbf{c}) \rangle \equiv \int d\mathbf{c} h^*(\mathbf{c}) g(\mathbf{c}), \quad (70)$$

the start denoting complex conjugated. The deviations ρ and θ can be expressed in terms of $\delta\chi$ as

$$\rho = \int d\mathbf{c} \delta\chi(\mathbf{c}, s), \quad \theta(s) = \int d\mathbf{c} \left(\frac{2}{d} c^2 - 1 \right) \delta\chi(\mathbf{c}, s), \quad (71)$$

and then

$$2\rho + \theta(s) = \int d\mathbf{c} \left(\frac{2}{d} c^2 + 1 \right) \delta\chi(\mathbf{c}, s) \equiv \langle \bar{\xi}_2(c) | \delta\chi(\mathbf{c}, s) \rangle, \quad (72)$$

where we have introduced

$$\bar{\xi}_2(c) \equiv \frac{2}{d} c^2 + 1. \quad (73)$$

By taking the scalar product with $\bar{\xi}_2$ in Eq. (45), it is obtained

$$\frac{d}{ds} \langle \bar{\xi}_2(c) | \delta\chi(\mathbf{c}, s) \rangle = \langle \bar{\xi}_2(c) | \Lambda(\mathbf{c}) \delta\chi(\mathbf{c}, s) \rangle. \quad (74)$$

Comparing this equation with the evolution equation for θ , Eq. (66), it is seen that, for $\delta\chi$ belonging to the biparametric family of functions of time-dependent USF states that are closed to the stationary USF state, $\langle \bar{\xi}_2(c) | \Lambda(\mathbf{c}) \delta\chi(\mathbf{c}, s) \rangle = -\gamma \langle \bar{\xi}_2(c) | \delta\chi(\mathbf{c}, s) \rangle$. Below it will be discussed while it is consistent to consider the approximation

$$\langle \bar{\xi}_2(c) | \Lambda(\mathbf{c}) g(\mathbf{c}) \rangle \approx -\gamma \langle \bar{\xi}_2(c) | g(\mathbf{c}) \rangle, \quad (75)$$

for any function, $g(\mathbf{c})$. This is, basically, the approximation that allows calculating the fluctuations of the total energy in [9, 29, 30]. Let us also mention that, in the free-cooling case, the equivalent of Eq. (75) is an exact property for Maxwell molecules [35].

Actually, it will be shown that Eq. (75), although consistent with linear hydrodynamics, is not consistent with the equation for ϕ , Eq. (49). To start with, let us see that some velocity moments of ϕ can be exactly related to velocity moments of the one-particle distribution, χ . As the total number of particles, N , does not fluctuate, it is evident that

$$\langle \delta N(t) \delta \mathcal{A}(t) \rangle = 0, \quad (76)$$

for any fluctuating quantity, \mathcal{A} . If, in addition, \mathcal{A} can be expressed as in Eq. (52), we have

$$\langle \delta N \delta \mathcal{A}(t) \rangle = N v_s^\beta \left[\int d\mathbf{c} a(\mathbf{c}) \chi(\mathbf{c}) + \int d\mathbf{c}_1 \int d\mathbf{c}_2 a(\mathbf{c}_1) \phi(\mathbf{c}_1, \mathbf{c}_2) \right], \quad (77)$$

and it can be concluded that

$$\int d\mathbf{c}_1 \int d\mathbf{c}_2 a(\mathbf{c}_1) \phi(\mathbf{c}_1, \mathbf{c}_2) = - \int d\mathbf{c} a(\mathbf{c}) \chi(\mathbf{c}), \quad (78)$$

for any homogeneous function, $a(\mathbf{c})$, of degree β . With this property we can easily calculate the component

$$\langle \bar{\xi}_2(c_1) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle = - \int d\mathbf{c} \left(\frac{2c^2}{d} + 1 \right) \chi(\mathbf{c}) = -2. \quad (79)$$

On the other hand, the integral can also be evaluated by taking the scalar product with $\bar{\xi}_2$ in the equation for ϕ , Eq. (49), obtaining

$$\langle \bar{\xi}_2(c_1) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle = \frac{1}{\gamma} \int d\mathbf{c}_1 \int d\mathbf{c}_2 \left(\frac{2c^2}{d} + 1 \right) \tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2) \chi(\mathbf{c}_1) \chi(\mathbf{c}_2) = -\frac{\tilde{\xi}_s}{\gamma}, \quad (80)$$

where the expression of $\tilde{\xi}$, Eq. (69), has been used, and we have introduced the notation $\tilde{\xi}_s \equiv \tilde{\xi}(\tilde{a}_s)$. Then, it follows that the approximation (75) is not consistent with the equation for the correlation function, Eq. (49), because it predicts a different result for $\langle \bar{\xi}_2(c_1) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle$ than the exact one giving by Eq. (79). Moreover, the approximation is not even valid in the elastic limit since $\gamma \sim \tilde{\xi}$ in that limit. In fact, when $\langle \delta E^2 \rangle$ is calculated using the approximate expression of $\langle c_1^2 c_2^2 | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle$ (evaluated using Eq. (75)), the obtained result does not agree with the results of [18] even for $\alpha \rightarrow 1$. Of course, this is not surprising, since the approximation is not valid in the elastic limit either.

C. A consistent approximation

The previous result is clearly unsatisfactory, and it would be desirable to find a kind of approximation that would be consistent with both linear hydrodynamics and the equation for the correlation function. Let us also note that the linearized Boltzmann operator, Λ , given by Eq. (46), contains a term of the form $c_y \partial / \partial c_x$ which mixes the subspace generated by c^2 with $c_x c_y$. Then, it can not be expected that $\langle \bar{\xi}_2 | \Lambda \approx -\gamma \langle \bar{\xi}_2 |$ be a good approximation in general. In fact, the operator $c_y \partial / \partial c_x$ leaves invariant the 4-dimensional subspace generated by $\{1, c^2, c_x c_y, c_y^2\}$ and, for Maxwell molecules, the left eigenfunctions of Λ are linear combination of these 4 functions [36]. With this in mind, we will search a generalization of approximation (75) taking as a possible candidate for $\bar{\xi}_2$ a function in the subspace generated by $\{1, c^2, c_x c_y, c_y^2\}$. To identify it, we consider the evolution equations for the homogeneous pressure tensor components of references [22, 24]

$$\frac{\partial T_H}{\partial t} + \zeta_H T_H + \frac{2a}{dn_H} P_{xy,H} = 0, \quad (81)$$

$$\frac{\partial P_{xy,H}}{\partial t} + (\beta \nu_H + \zeta_H) P_{xy,H} + a P_{yy,H} = 0, \quad (82)$$

$$\frac{\partial P_{yy,H}}{\partial t} + (\beta \nu_H + \zeta_H) P_{yy,H} - \beta n_H \nu_H T_H = 0, \quad (83)$$

where we have introduced the subindex H to remark that we are only considering homogeneous situations. The cooling rate can be expressed as

$$\zeta_H = \frac{v_H}{\lambda} \tilde{\zeta}_s, \quad \tilde{\zeta}_s = \frac{\sqrt{2} \pi^{(d-1)/2} (1 - \alpha^2)}{d \Gamma(d/2)}, \quad (84)$$

where $\tilde{\zeta}_s$ coincides with $\tilde{\zeta}(\tilde{a}_s)$ calculated in the Jenkins and Richman approximation to ϵ^2 order, ν_H is the collision frequency

$$\nu_H = \frac{v_H}{\lambda} z, \quad z = \frac{8 \pi^{(d-1)/2}}{\sqrt{2} (d+2) \Gamma(d/2)},$$

and β is a parameter to be specified later on.

Equations (81)-(83) admit a stationary solution. Defining the dimensionless components of the pressure tensor in the stationary state

$$\tilde{P}_{ij,s} \equiv \frac{P_{ij,s}}{n_s T_s}, \quad (85)$$

it is obtained [22, 24]

$$\tilde{P}_{xy,s} = -\frac{d \tilde{\zeta}_s}{2 \tilde{a}_s}, \quad \tilde{P}_{yy,s} = \frac{\beta}{\beta + \frac{\tilde{\zeta}_s}{z}}. \quad (86)$$

The dimensionless shear rate is

$$\tilde{a}_s = z \sqrt{\frac{d\tilde{\zeta}_s}{2z\beta}} \left(\beta + \frac{\tilde{\zeta}_s}{z} \right), \quad (87)$$

from which the stationary temperature can be evaluated through $v_s = \lambda a / \tilde{a}_s$. Note that all the expressions can be expressed in terms of $\tilde{\zeta}_s$ and β .

The set of equations (81)-(83), plus the equation for the total density that is trivial, can be linearized around the stationary state characterized by n_s , T_s and $P_{ij,s}$. Defining the dimensionless deviations of the pressure tensor as

$$\Pi_{ij} = \frac{P_{ij} - P_{ij,s}}{n_s T_s}, \quad (88)$$

we obtain the following set of linear equations

$$\frac{d}{ds} \mathbf{y}(s) + M \mathbf{y}(s) = \mathbf{0}, \quad (89)$$

for

$$\mathbf{y} = \begin{bmatrix} \rho(s) \\ \theta(s) \\ \Pi_{xy}(s) \\ \Pi_{yy}(s) \end{bmatrix}, \quad (90)$$

where we have introduced the matrix

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2\tilde{\zeta}_s & \frac{3}{2}\tilde{\zeta}_s & (\beta z + \tilde{\zeta}_s) \sqrt{\frac{2\tilde{\zeta}_s}{d\beta z}} & 0 \\ -\sqrt{\frac{d\tilde{\zeta}_s \beta z}{2}} & -\frac{1}{2} \sqrt{\frac{d\tilde{\zeta}_s \beta z}{2}} & \tilde{\zeta}_s + \beta z & (\beta z + \tilde{\zeta}_s) \sqrt{\frac{d\tilde{\zeta}_s}{2\beta z}} \\ -\beta z & -\beta z & 0 & \tilde{\zeta}_s + \beta z \end{bmatrix}, \quad (91)$$

that, again, is expressed in terms of $\tilde{\zeta}_s$ and β . Taking the explicit value of β evaluated in Grad's approximation [24]

$$\beta = \frac{1 + \alpha}{2} \left[1 - \frac{d-1}{2d} (1 - \alpha) \right], \quad (92)$$

the matrix is expressed uniquely in terms of the inelasticity, α . In this way, Eq. (89) is a set of linear differential equations for the deviations, \mathbf{y} , defined in Eq. (90), where all the coefficients of the matrix M are known functions of the coefficient of normal restitution, α .

This is the generalization of Eq. (66) that we were looking for. The eigenvalues, $\{\lambda_i\}_{i=1}^4$, and their corresponding left eigenfunctions of M , $\{\mathbf{v}_i\}_{i=1}^4$, fulfill

$$\mathbf{v}_i \cdot M = \lambda_i \mathbf{v}_i, \quad (93)$$

and can be calculated with Mathematica. As the expressions are very long, here we just write the expansion to ϵ^4 order for $d = 2$

$$\lambda_1 = 0, \quad \lambda_2 \approx \sqrt{\frac{\pi}{2}}\epsilon^2 - \frac{3}{4}\sqrt{\frac{\pi}{2}}\epsilon^4, \quad (94)$$

$$\lambda_3 \approx \left(\sqrt{2\pi} + \frac{1}{2}\sqrt{\frac{\pi}{2}}\epsilon^2 + \frac{1}{4}\sqrt{\frac{\pi}{2}}\epsilon^4 \right) - i \left(\sqrt{\pi}\epsilon + \frac{19}{64}\epsilon^3 \right), \quad \lambda_4 = \lambda_3^*. \quad (95)$$

The corresponding left eigenfunctions to the same order are

$$\mathbf{v}_1 = (1, 0, 0, 0), \quad (96)$$

$$\mathbf{v}_2 \approx \left(4 - \frac{7\epsilon^2}{2} + \frac{17\epsilon^4}{4}, 2 - \frac{11\epsilon^2}{4} + \frac{23\epsilon^4}{8}, -\sqrt{2}\epsilon + \frac{11\epsilon^3}{8\sqrt{2}}, \epsilon^2 \right), \quad (97)$$

$$\begin{aligned} \mathbf{v}_3 \approx & \left(-1 - \frac{\epsilon^2}{2} + \epsilon^4, -1 + \frac{\epsilon^2}{16} + \frac{67\epsilon^4}{128}, \frac{\epsilon}{4\sqrt{2}} + \frac{19\epsilon^3}{64\sqrt{2}}, 1 \right) \\ & - i \left(\frac{3\epsilon^3}{2\sqrt{2}}, \frac{\epsilon}{2\sqrt{2}} + \frac{75\epsilon^3}{128\sqrt{2}}, 1 - \frac{\epsilon^2}{64} - \frac{1841\epsilon^4}{8192}, 0 \right), \end{aligned} \quad (98)$$

$$\mathbf{v}_4 = \mathbf{v}_3^*. \quad (99)$$

Let us remark that, as Eq. (81) was the starting point for the derivation of Eq. (66), λ_2 can be expressed in a way similar to γ ,

$$\lambda_2 = \frac{\tilde{\zeta}(\tilde{a}_s)}{2} - \frac{\tilde{a}_s^2}{d} \frac{d\tilde{P}_{xy}}{d\tilde{a}}(\tilde{a}_s). \quad (100)$$

Here we do not have the $\frac{d\tilde{\zeta}}{d\tilde{a}}$ contribution since it was neglected from the very beginning.

With the aid of the left eigenfunctions, the normal modes of Eq. (89) can be easily written as

$$\Xi_j = \mathbf{v}_j \cdot \mathbf{y} = v_{j1}\rho + v_{j2}\theta + v_{j3}\Pi_{xy} + v_{j4}\Pi_{yy}, \quad (101)$$

where v_{ji} is the i -th component of \mathbf{v}_j . Now, we can identify the functions, $\{\bar{\xi}_i(\mathbf{c})\}_{i=1}^4$,

$$\bar{\xi}_i(\mathbf{c}) = \xi_{i1} + \xi_{i2}c^2 + \xi_{i3}c_x c_y + \xi_{i4}c_y^2, \quad (102)$$

such that

$$\langle \bar{\xi}_j(\mathbf{c}) | \delta\chi(\mathbf{c}) \rangle = \Xi_j. \quad (103)$$

Taking into account Eq. (71) and

$$\Pi_{ij}(s) = 2 \int d\mathbf{c} c_i c_j \delta\chi(\mathbf{c}, s), \quad (104)$$

we can identify

$$\bar{\xi}_1(c) = 1, \quad (105)$$

$$\bar{\xi}_2(\mathbf{c}) = (v_{21} - v_{22}) + \frac{2}{d} v_{22} c^2 + 2v_{23} c_x c_y + 2v_{24} c_y^2, \quad (106)$$

$$\bar{\xi}_3(\mathbf{c}) = (v_{31} - v_{32}) + \frac{2}{d} v_{32} c^2 + 2v_{33} c_x c_y + 2c_y^2, \quad (107)$$

$$\bar{\xi}_4(\mathbf{c}) = \bar{\xi}_3^*(\mathbf{c}). \quad (108)$$

Note that while the coefficients $\{v_{2j}\}_{j=1}^4$ are real, $\{v_{3j}\}_{j=1}^3$ have an imaginary part. Then the real and imaginary part of λ_3 and $\bar{\xi}_3$ are introduced through

$$\lambda_3 = \lambda_3^R + \imath \lambda_3^I, \quad (109)$$

$$\bar{\xi}_3(\mathbf{c}) = \bar{\xi}_3^R(\mathbf{c}) + \imath \bar{\xi}_3^I(\mathbf{c}). \quad (110)$$

In Appendix B it is shown that the approximation

$$\langle \bar{\xi}_i(\mathbf{c}) | \Lambda(\mathbf{c}) g(\mathbf{c}) \rangle \approx -\lambda_i \langle \bar{\xi}_i(\mathbf{c}) | g(\mathbf{c}) \rangle, \quad i = 2, 3, 4. \quad (111)$$

is consistent with the equation for the correlation function, Eq. (49). Taking the scalar product with $\{\bar{\xi}_i\}_{i=1}^3$ in Eq. (49) an identity is obtained. Therefore, in contrast with approximation (75), the approximation given by Eq. (111) is fully consistent, i.e. it is compatible with both linear hydrodynamics and the equation for the two-particle correlations.

To summarize, we have identified four modes. The first one (with the null eigenvalue) is trivial because is the one associated to the total number of particles. The second eigenvalue, $\lambda_2 = \gamma$, vanishes in the elastic limit and is the one associated with the slowest excitations (at least in the elastic limit). For this reason, the second mode, Ξ_2 , will be referred to as the hydrodynamic mode, in the following. The last two modes (one is the complex conjugate of the other) decay faster and will be called kinetic modes. Let us note that, although we have extended the number of fields to describe the excitations of the system, the number of slow modes remains the same (i.e. we have not adopted a kind of extended hydrodynamics approach as it could seem at first sight). Of course, these results are consistent with the ones of section IV A. We obtain the same eigenvalue and, although the associated eigenfunctions

are different, both modes are equivalent in the proper subspace. The differences in the modes are not important at the level of macroscopic hydrodynamics, but they are crucial at the level of two-particle correlations and, therefore, to identify the correct fluctuating hydrodynamic equations [18].

Let us evaluate the fluctuations of the total energy using the approximation given by (111). This can be done by taking the scalar products with $\langle \bar{\xi}_i(\mathbf{c}_1) \bar{\xi}_j(\mathbf{c}_2) |$ in the Eq. (49) for $i, j = 2, 3, 4$, but, in contrast to the previous cases, these fluctuations are coupled to the ones of the pressure tensor. Nevertheless, we will see that this coupling disappears in the elastic limit (here we will restrict ourselves to $d = 2$). In effect, multiplying Eq. (49) with $\langle \bar{\xi}_2(\mathbf{c}_1) \bar{\xi}_2(\mathbf{c}_2) |$ it is obtained

$$2\gamma \langle \bar{\xi}_2(\mathbf{c}_1) \bar{\xi}_2(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle = -\langle \bar{\xi}_2(\mathbf{c}_1) \bar{\xi}_2(\mathbf{c}_2) | \tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2) \chi(\mathbf{c}_1) \chi(\mathbf{c}_2) \rangle, \quad (112)$$

where approximation (111) has been used. For $d = 2$ and to leading order (ϵ^2 order in this case), we have

$$\begin{aligned} \gamma \langle \bar{\xi}_2(\mathbf{c}_1) \bar{\xi}_2(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle &\approx \sqrt{\frac{\pi}{2}} \epsilon^2 \langle (2 + 2c_1^2)(2 + 2c_2^2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle \\ &= \sqrt{\frac{\pi}{2}} \epsilon^2 [4 \langle c_1^2 c_2^2 | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle - 12]. \end{aligned} \quad (113)$$

The relation (78) has been used to evaluate

$$\int d\mathbf{c}_1 \phi(\mathbf{c}_1, \mathbf{c}_2) = \int d\mathbf{c}_1 c_i^2 \phi(\mathbf{c}_1, \mathbf{c}_2) = -1. \quad (114)$$

The right hand side of Eq. (112) is evaluated in Appendix C using the ϵ expansion of the Jenkins and Richman approximation, obtaining

$$\langle \bar{\xi}_2(\mathbf{c}_1) \bar{\xi}_2(\mathbf{c}_2) | \tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2) \chi(\mathbf{c}_1) \chi(\mathbf{c}_2) \rangle \approx -16\sqrt{2\pi}\epsilon^2. \quad (115)$$

By introducing Eqs. (113) and (115) into Eq. (112), we have

$$\lim_{\alpha \rightarrow 1} \langle c_1^2 c_2^2 | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle = -1. \quad (116)$$

Finally, taking into account Eq. (60), we can calculate the elastic limit of

$$N \frac{\langle \delta \mathcal{E}^2 \rangle}{\langle \mathcal{E} \rangle^2} = \left[\int d\mathbf{c} c^4 \chi(\mathbf{c}) + \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_1^2 c_2^2 \phi(\mathbf{c}_1, \mathbf{c}_2) \right] \rightarrow 1, \quad (117)$$

consistently with the results of [18].

V. FLUCTUATIONS OF THE RELEVANT GLOBAL QUANTITIES

The structure of the modes derived above implies a coupling between the fluctuations of the total energy and the fluctuations of the pressure tensor for finite ϵ . In this section, all these cross correlations will be evaluated. The fluctuating total pressure tensor is defined as

$$\mathcal{P}_{ij}(t) \equiv m \int d\mathbf{r} \int d\mathbf{v} V_i V_j F_1(x, t), \quad (118)$$

and its deviation can be written in the form indicated in Eq. (53). The correlations between $\delta\mathcal{E}$ and $\delta\mathcal{P}_{ij}$ can be calculated with the aid of Eq. (58), obtaining

$$\langle \delta\mathcal{E}(t) \delta\mathcal{P}_{ij}(t) \rangle = \frac{m^2}{2} N v_s^4 \left[\int d\mathbf{c} c^2 c_i c_j \chi(\mathbf{c}) + \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_1^2 c_{2i} c_{2j} \phi(\mathbf{c}_1, \mathbf{c}_2) \right]. \quad (119)$$

Analogously, it is

$$\langle \delta\mathcal{P}_{ij}(t) \delta\mathcal{P}_{nm}(t) \rangle = m^2 N v_s^4 \left[\int d\mathbf{c} c_i c_j c_n c_m \chi(\mathbf{c}) + \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1i} c_{1j} c_{2n} c_{2m} \phi(\mathbf{c}_1, \mathbf{c}_2) \right]. \quad (120)$$

This expression involves the first velocity moments of the correlation function, ϕ . It is convenient to introduce the following notation

$$\mathbf{b}(\mathbf{c}) = \begin{bmatrix} 1 \\ c^2 \\ c_x c_y \\ c_y^2 \end{bmatrix}, \quad (121)$$

allowing to express the moments in the following matrix form

$$C_{ij} = \int d\mathbf{c}_1 \int d\mathbf{c}_2 b_i(\mathbf{c}_1) b_j(\mathbf{c}_2) \phi(\mathbf{c}_1, \mathbf{c}_2), \quad (122)$$

that is trivially symmetric, i.e. $C_{ij} = C_{ji}$. In fact, the moments $\{C_{1j}\}_{j=1}^4$ can be easily calculated due to the conservation of the total number of particles. Taking into account Eq. (78), we get

$$C_{11} = -1, \quad C_{12} = -1, \quad C_{13} = -\frac{1}{2} \tilde{P}_{xy,s}, \quad C_{14} = -\frac{1}{2} \tilde{P}_{yy,s}. \quad (123)$$

To calculate the other C_{ij} the scalar product $\langle \bar{\xi}_i(\mathbf{c}_1) \bar{\xi}_j(\mathbf{c}_2) |$ is taken in Eq. (49) and the approximation (111) introduced, obtaining

$$(\lambda_i + \lambda_j) \langle \bar{\xi}_i(\mathbf{c}_1) \bar{\xi}_j(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle = \langle \bar{\xi}_i(\mathbf{c}_1) \bar{\xi}_j(\mathbf{c}_2) | \tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2) \chi(\mathbf{c}_1) \chi(\mathbf{c}_2) \rangle, \quad i, j = 2, 3, 4. \quad (124)$$

Actually, there are only 6 independent equations, because of the relation between the third and fourth modes. As the scalar products $\langle \bar{\xi}_i(\mathbf{c}_1) \bar{\xi}_j(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle$ (see Eq. (102) and (122)) can be written in terms of the C_{ij} coefficients through

$$\langle \bar{\xi}_i(\mathbf{c}_1) \bar{\xi}_j(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle = \sum_{l=1}^4 \xi_{il} \xi_{jl} C_{ll} + \sum_{k>l=1}^4 (\xi_{ik} \xi_{jl} + \xi_{il} \xi_{jk}) C_{kl}, \quad (125)$$

Eq. (124) define a linear system of six equations for the six unknown coefficients $\{C_{22}, C_{23}, C_{24}, C_{33}, C_{34}, C_{44}\}$ (remember that $\{C_{1j}\}_{j=1}^4$ are known).

The calculation leading to the expressions of the coefficients C_{ij} are detailed in Appendix C. Since the $\langle \bar{\xi}_i(\mathbf{c}_1) \bar{\xi}_j(\mathbf{c}_2) | \tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2) \chi(\mathbf{c}_1) \chi(\mathbf{c}_2) \rangle$ are evaluated using the Jenkins and Richman distribution function for $d = 2$, in the following all the results are restricted to this dimension. To order ϵ^2 the obtained expressions are

$$\begin{aligned} C_{22} &= -1 + \frac{27}{32} \epsilon^2, & C_{23} &= \frac{5}{8\sqrt{2}} \epsilon, & C_{24} &= -\frac{1}{2} + \frac{51}{64} \epsilon^2, \\ C_{33} &= -\frac{23}{256} \epsilon^2, & C_{34} &= \frac{5}{16\sqrt{2}} \epsilon, & C_{44} &= -\frac{1}{4} + \frac{145}{256} \epsilon^2. \end{aligned} \quad (126)$$

The one-particle averages that appear in the equations of the fluctuations, $\langle b_i(\mathbf{c}) b_j(\mathbf{c}) | \chi(\mathbf{c}) \rangle$, can be calculated in the same approximation, obtaining

$$\begin{aligned} \langle c^4 | \chi(\mathbf{c}) \rangle &= 2 + \frac{1}{2} \epsilon^2, & \langle c^2 c_x c_y | \chi(\mathbf{c}) \rangle &= -\frac{3}{2\sqrt{2}} \epsilon, & \langle c^2 c_y^2 | \chi(\mathbf{c}) \rangle &= 1 - \frac{1}{2} \epsilon^2, \\ \langle c_x^2 c_y^2 | \chi(\mathbf{c}) \rangle &= \frac{1}{4} (1 + \epsilon^2), & \langle c_x c_y^3 | \chi(\mathbf{c}) \rangle &= -\frac{3}{4\sqrt{2}} \epsilon, & \langle c_y^4 | \chi(\mathbf{c}) \rangle &= \frac{3}{4} (1 - \epsilon^2). \end{aligned} \quad (127)$$

To express the final result in a compact notation it is useful to introduce the matrix elements

$$B_{ij}(0) \equiv \int d\mathbf{c}_1 \int d\mathbf{c}_2 b_i(\mathbf{c}_1) b_j(\mathbf{c}_2) \psi(\mathbf{c}_1, \mathbf{c}_2, 0) = \langle b_i(\mathbf{c}) b_j(\mathbf{c}) | \chi(\mathbf{c}) \rangle + C_{ij}. \quad (128)$$

By substituting Eqs. (126) and (127) into the equation above, the expansion to second order in ϵ of $B(0)$ is obtained,

$$B(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 + \frac{199\epsilon^2}{64} & -\frac{7\epsilon}{8\sqrt{2}} & \frac{1}{2} + \frac{19\epsilon^2}{64} \\ 0 & -\frac{7\epsilon}{8\sqrt{2}} & \frac{1}{4} + \frac{41\epsilon^2}{256} & -\frac{7\epsilon}{16\sqrt{2}} \\ 0 & \frac{1}{2} + \frac{19\epsilon^2}{64} & -\frac{7\epsilon}{16\sqrt{2}} & \frac{1}{2} - \frac{47\epsilon^2}{256} \end{bmatrix}. \quad (129)$$

Finally, taking into account Eqs. (128), (60), (119), and (120), we can express all the correlation functions in terms of $B_{ij}(0)$,

$$\begin{aligned}\langle \delta \mathcal{E}^2(t) \rangle &= \frac{m^2}{4} N v_s^4 B_{22}(0), & \langle \delta \mathcal{E}(t) \delta \mathcal{P}_{xy}(t) \rangle &= \frac{m^2}{2} N v_s^4 B_{23}(0), \\ \langle \delta \mathcal{E}(t) \delta \mathcal{P}_{yy}(t) \rangle &= \frac{m^2}{2} N v_s^4 B_{24}(0), & \langle \delta \mathcal{P}_{xy}^2(t) \rangle &= m^2 N v_s^4 B_{33}(0), \\ \langle \delta \mathcal{P}_{xy}(t) \delta \mathcal{P}_{yy}(t) \rangle &= m^2 N v_s^4 B_{34}(0), & \langle \delta \mathcal{P}_{yy}^2(t) \rangle &= m^2 N v_s^4 B_{44}(0).\end{aligned}\quad (130)$$

It is worth to remark that, although the system has been solved consistently to ϵ^2 order, the expressions for the correlation functions are not the exact power expansion of the correlation functions. This is so because the Jenkins and Richman approximation to ϵ^2 order is not the exact expansion of the distribution [21].

Finally, let us calculate the two-time correlation functions between the already considered global quantities. Using Eq. (57) we arrive to the generalization of Eqs. (60), (119) and (120) for two times

$$\langle \delta \mathcal{E}(t) \delta \mathcal{E}(t') \rangle = \frac{m^2}{4} N v_s^4 \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_1^2 c_2^2 \psi(\mathbf{c}_1, \mathbf{c}_2; s - s'), \quad (131)$$

$$\langle \delta \mathcal{E}(t) \delta \mathcal{P}_{ij}(t') \rangle = \frac{m^2}{2} N v_s^4 \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_1^2 c_{2i} c_{2j} \psi(\mathbf{c}_1, \mathbf{c}_2, s - s'), \quad (132)$$

$$\langle \delta \mathcal{P}_{ij}(t) \delta \mathcal{E}(t') \rangle = \frac{m^2}{2} N v_s^4 \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1i} c_{1j} c_2^2 \psi(\mathbf{c}_1, \mathbf{c}_2, s - s'), \quad (133)$$

$$\langle \delta \mathcal{P}_{ij}(t) \delta \mathcal{P}_{nm}(t') \rangle = m^2 N v_s^4 \int d\mathbf{c}_1 \int d\mathbf{c}_2 c_{1i} c_{1j} c_{2n} c_{2m} \psi(\mathbf{c}_1, \mathbf{c}_2, s - s'). \quad (134)$$

Again, it is convenient to define the matrix elements

$$B_{ij}(s) \equiv \int d\mathbf{c}_1 \int d\mathbf{c}_2 b_i(\mathbf{c}_1) b_j(\mathbf{c}_2) \psi(\mathbf{c}_1, \mathbf{c}_2, s). \quad (135)$$

Inserting the formal expression of $\psi(s)$

$$\psi(\mathbf{c}_1, \mathbf{c}_2, s) = e^{s\Lambda(\mathbf{c}_1)} [\chi(\mathbf{c}_1) \delta(\mathbf{c}_{12}) + \phi(\mathbf{c}_1, \mathbf{c}_2)], \quad (136)$$

into the above equations, and taking into account that the functions $\{b_i(\mathbf{c})\}_{i=1}^4$ can be written in terms of the functions $\{\bar{\xi}_i(\mathbf{c})\}_{i=1}^4$, the correlation functions can be evaluated explicitly by using the approximation (111), with the result

$$B_{ij}(s) = \sum_{l=1}^4 \sum_{k=1}^4 Q_{ik}^{-1} Q_{kl} e^{\lambda_k s} B_{lj}(0), \quad s > 0. \quad (137)$$

Here we have introduced the matrix Q

$$\begin{bmatrix} \bar{\xi}_1(\mathbf{c}) \\ \bar{\xi}_2(\mathbf{c}) \\ \bar{\xi}_3(\mathbf{c}) \\ \bar{\xi}_4(\mathbf{c}) \end{bmatrix} = Q\mathbf{b}(\mathbf{c}), \quad (138)$$

and its inverse, Q^{-1} , that can be identified through Eq. (102). To order ϵ^2 we have

$$Q = \begin{bmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} \\ \xi_{21} & \xi_{22} & \xi_{23} & \xi_{24} \\ \xi_{31} & \xi_{32} & \xi_{33} & \xi_{34} \\ \xi_{41} & \xi_{42} & \xi_{43} & \xi_{44} \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 - \frac{3\epsilon^2}{4} & 2 - \frac{11\epsilon^2}{4} & -2\sqrt{2}\epsilon & 2\epsilon^2 \\ -\frac{9\epsilon^2}{16} & -1 + \frac{\epsilon^2}{16} - \imath\frac{\epsilon}{2\sqrt{2}} & \frac{\epsilon}{2\sqrt{2}} - 2\imath\left(1 - \frac{\epsilon^2}{64}\right) & 2 \\ -\frac{9\epsilon^2}{16} & -1 + \frac{\epsilon^2}{16} + \imath\frac{\epsilon}{2\sqrt{2}} & \frac{\epsilon}{2\sqrt{2}} + 2\imath\left(1 - \frac{\epsilon^2}{64}\right) & 2 \end{bmatrix}. \quad (139)$$

Obviously, the correlation functions given by (137) fulfill the initial conditions. Moreover, they are a linear combination of the two modes λ_2 and λ_3 ($\lambda_4 = \lambda_3^*$). The complete expressions for all the correlation functions are very lengthy and here we only write explicitly the expressions for the two-time autocorrelation function of the energy and pressure tensor,

$$\langle \delta\mathcal{E}(t)\delta\mathcal{E}(0) \rangle = \frac{m^2}{4}Nv_s^4 B_{22}(s), \quad \langle \delta\mathcal{P}_{xy}(t)\delta\mathcal{P}_{xy}(0) \rangle = m^2Nv_s^4 B_{33}(s), \quad (140)$$

with $B_{22}(s)$ and $B_{33}(s)$ given by Eq. (137), i.e.

$$B_{22}(s) = \left(1 + \frac{225\epsilon^2}{128}\right) e^{-\lambda_2 s} - \frac{5\epsilon^2}{8} \cos(\lambda_3^I s) e^{-\lambda_3^R s}, \quad (141)$$

$$B_{33}(s) = \left[\left(\frac{1}{4} - \frac{3\epsilon^2}{256}\right) \cos(\lambda_3^I s) + \frac{\sqrt{2}\epsilon}{32} \sin(\lambda_3^I s)\right] e^{-\lambda_3^R s} + \frac{11\epsilon^2}{64} e^{-\lambda_2 s}. \quad (142)$$

We see that both functions have a hydrodynamic and a kinetic part. Nevertheless, the main contribution of $B_{22}(s)$ is the hydrodynamic one (the kinetic part is of order ϵ^2), while the opposite occurs with $B_{33}(s)$.

VI. SIMULATION RESULTS

We have performed Molecular Dynamics (MD) simulations of a two dimensional system of $N = 2000$ inelastic hard disks of mass m and diameter σ , in a square box of side L , corresponding to a number density $n_s = 0.02\sigma^{-2}$. To generate the stationary USF state, Lees-Edwards boundary conditions [27] in the y -direction and periodic boundary conditions

in the x -direction have been used. Once the steady state is reached, we have measured all the quantities studied in the previous section. The reported values have been averaged over 300 trajectories, and also on time, over a period of about 150 collisions per particle. This has been done for different values of the inelasticity, α . The shear rate was in all cases $a = 6.32 \times 10^{-3}(T(0)/m)^{1/2}\sigma^{-1}$, where $T(0)$ is the initial temperature.

In Fig. 1 we plot the quantity $B_{22}(0)$ as a function of the inelasticity. The symbols are the simulation results and the solid line the theoretical prediction given by Eq. (129). We have also plotted the theoretical prediction of the fluctuating hydrodynamic approach (dashed line) and the improved one (dotted-line) taking into account rheological effects in the viscosity [18]. As it is shown in the figure, the last one is very close to the prediction given by (129). In Fig. 2 we plot the rest of matrix elements of B as a function of α . The solid lines are the theoretical predictions and the symbols are the simulation results. While the agreement is very good for $B_{23}(0)$, $B_{33}(0)$ and $B_{34}(0)$, we find some discrepancies for $B_{22}(0)$, $B_{24}(0)$ and $B_{44}(0)$ as the inelasticity increases.

As the $B_{ij}(0)$ coefficients have two components, the one-particle component and the correlation function component, we have measured the one-particle moments implied in order to study the origin of the discrepancies. In Fig. 3, we plot $\langle c^4 \rangle$ and $\langle c^2 c_y^2 \rangle$. As it is seen in the figure, there are important differences between the simulation results (points) and the Jenkins and Richman approximation (solid line). We also plot the theoretical prediction of the moments to ϵ^2 order using the BGK model [22] finding a remarkably better agreement. Its explicit expressions are

$$\langle c^4 \rangle_{BGK} = 2 + \epsilon^2, \quad \langle c^2 c_y^2 \rangle_{BGK} = 1 - \frac{\epsilon^2}{4}. \quad (143)$$

The rest of moments are accurately described by the Jenkins and Richman approximation. In fact, they coincide with the BGK ones apart from $\langle c_x^2 c_y^2 \rangle$, for which the Jenkins and Richman approximation goes better than the BGK prediction. In Fig. 4 we plot $B_{22}(0)$ and $B_{24}(0)$ using the one-particle moments of the BGK model, finding that this increases considerably the agreement with the simulation results. Hence, we can conclude that the agreement between the simulation results and the theoretical predictions (considering the most accurate expression for the one-particle moments) is excellent for all the coefficients for the range of inelasticities considered. The only exception is B_{44} for which the agreement is moderately good.

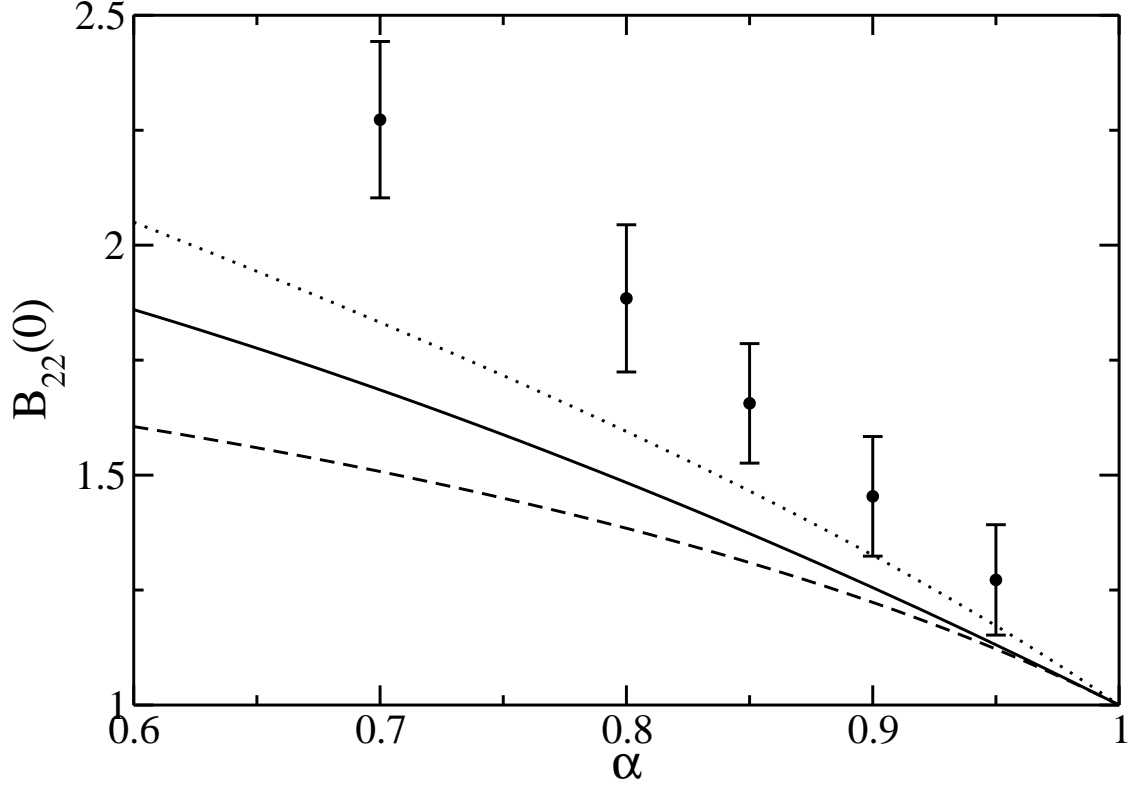


Figure 1. Dimensionless matrix element $B_{22}(0)$ as function of the restitution coefficient, α , for a system of $N = 2000$ hard disks. The symbols (dots) are the simulation results, the solid line is the theoretical prediction given by Eq. (129), the dashed line is the prediction using fluctuating hydrodynamics, and the dotted line the improved prediction given in [18].

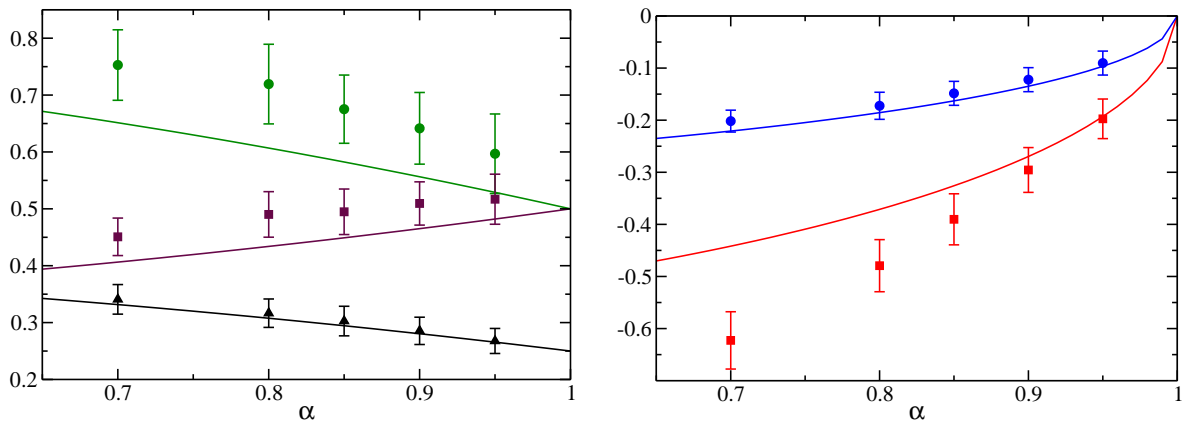


Figure 2. (Color on line) Dimensionless matrix elements of $B(0)$. The symbols are the simulation data and the solid lines correspond to the expansion to second order in ϵ given in Eq. (129). In the left figure, the circles, squares and triangles correspond to $B_{24}(0)$, $B_{44}(0)$ and $B_{33}(0)$ respectively. In the right figure, the circles and squares correspond to $B_{34}(0)$ and $B_{23}(0)$ respectively.

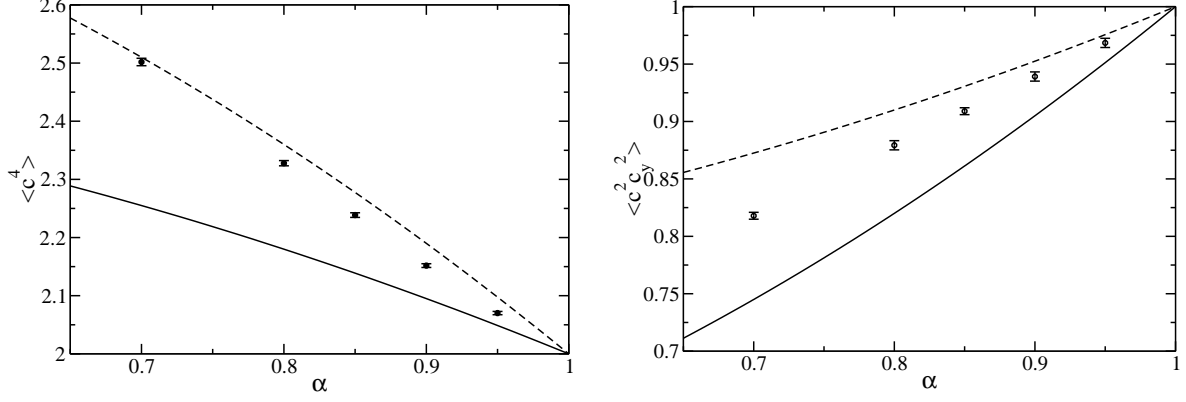


Figure 3. One-particle averages $\langle c^4 \rangle$ (left) and $\langle c^2 c_y^2 \rangle$ (right). The simulation data (symbols) are compared to the predictions of Jenkins and Richman [20] (solid line) and to the BGK model [22] (dashed line).

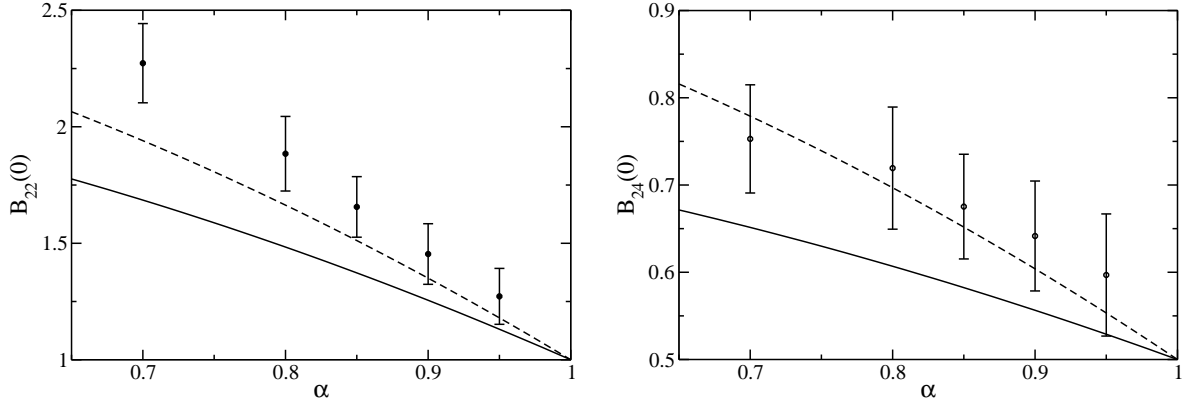


Figure 4. Comparison of the simulation results for $B_{22}(0)$ and $B_{24}(0)$ with the theoretical predictions using the Jenkins and Richman approximation [20] (solid lines) or the BGK model [22] (dashed line) for the one-particle moments.

Finally, we have also measured the two-time correlation functions. Fig. 5 shows the evolution of $B_{22}(s)/B_{22}(0)$, for systems with $\alpha = 0.80$ (left) and $\alpha = 0.90$ (right). In Fig. 6 the decay of $B_{33}(s)/B_{33}(0)$ has been plotted for the same values of the inelasticity. As the correlation functions are a combination of two exponentials, it is difficult to make a detail comparison between the theoretical prediction and the simulation results. Nevertheless, it is observed that $B_{33}(s)$ decays faster than $B_{22}(s)$, as predicted by Eqs. (141) and (142). $B_{33}(s)$ has a hydrodynamic part that is of order ϵ^2 , while the kinetic part is of order unity (the contrary occurs for $B_{22}(s)$). We have also seen that, in the long-time limit, both correlation functions decay in the same way (with the hydrodynamic mode). In order to see the long

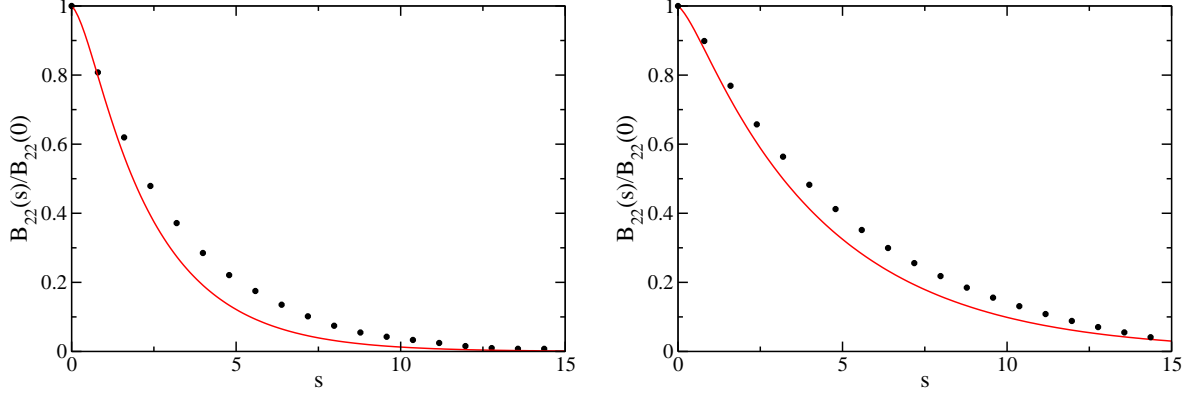


Figure 5. (Color on line) Decay of $B_{22}(s)/B_{22}(0)$ for a system with $\alpha = 0.80$ (left) and $\alpha = 0.90$ (right). The solid lines (red) are the predictions given by Eq. (141), and the symbols are the simulation results.

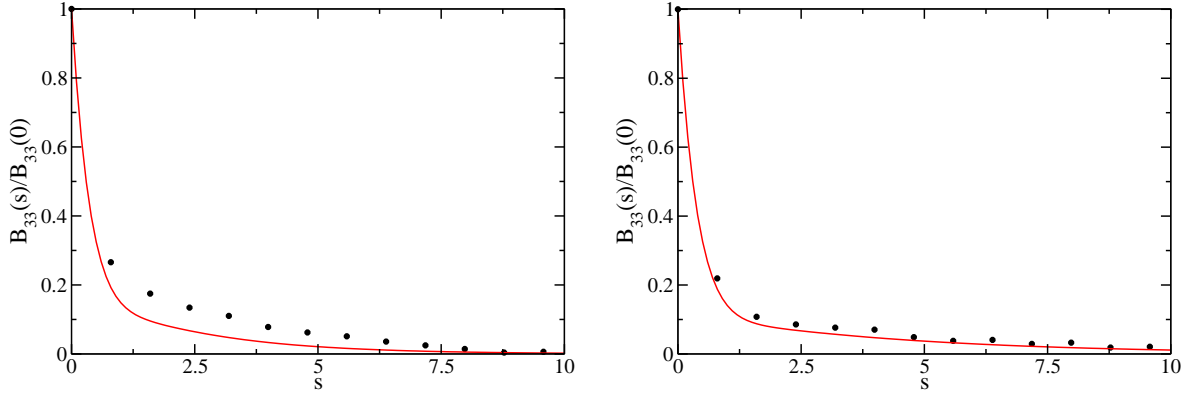


Figure 6. (Color on line) Decay of $B_{33}(s)/B_{33}(0)$ for a system with $\alpha = 0.80$ (left) and $\alpha = 0.90$ (right). The solid lines (red) are the predictions given by Eq. (142), and the symbols are the simulation results.

time behavior of the functions, they have been plotted in a logarithmic scale. This is done in Fig. 7 for a system with $\alpha = 0.90$, where it is seen that the slopes become the same when the time s is large enough.

VII. CONCLUSION AND DISCUSSION

In this paper, we have studied the fluctuations of the total internal energy of a granular gas in the stationary USF state. Using the approximation given by Eq. (111), it has been shown that the fluctuations of the total internal energy are coupled to the fluctuations of the several components of the total pressure tensor. The approximation is fully consistent with

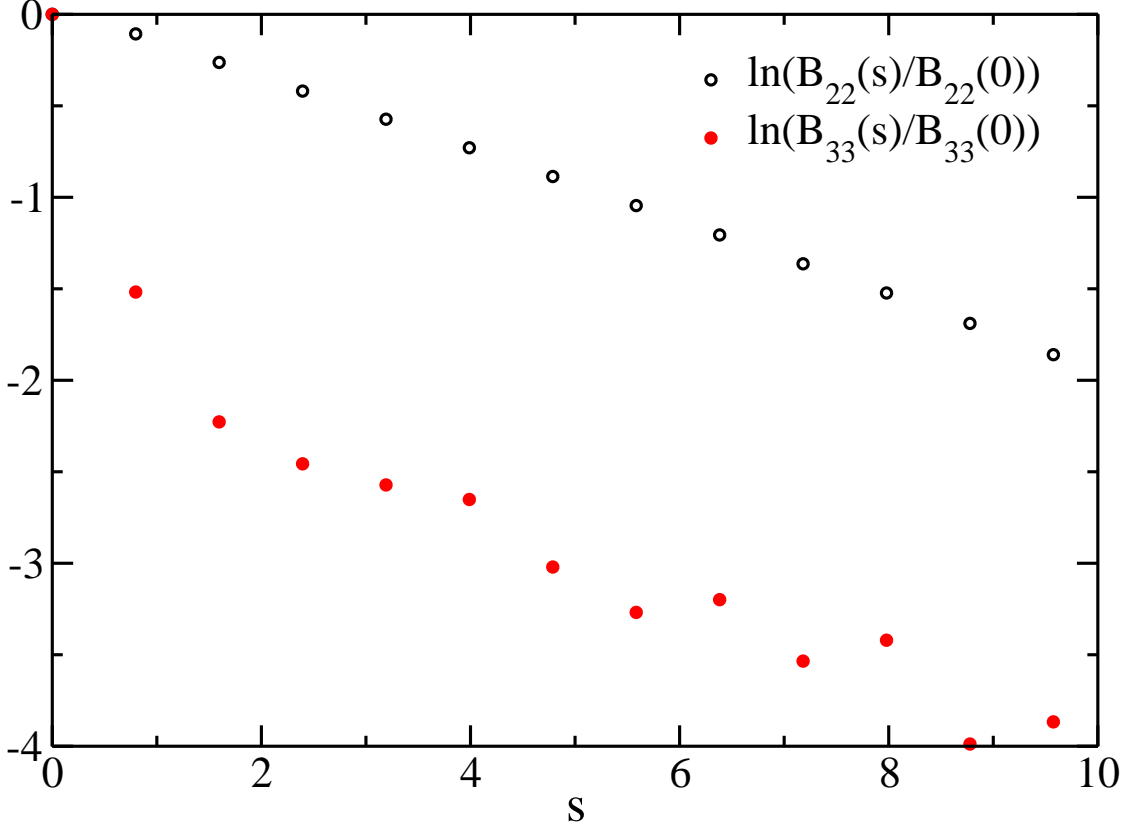


Figure 7. Comparison of the decay in the long time limit of the correlations $B_{22}(s)/B_{22}(0)$ and $B_{33}(s)/B_{33}(0)$ for a system with $\alpha = 0.90$.

the kinetic equation for the correlation function and, in principle, is not limited to small inelasticities. One of the main results of the paper is the closed system of equations given in (124), for the first six moments of the correlation function. With them, one can calculate all the possible one-time correlations of the total internal energy and the different components of the total pressure tensor. The system depends on several complex moments of the one-particle distribution function in the stationary USF state. Since this distribution function is not known exactly, the Jenkins and Richman distribution has been used to ϵ^2 order. For $d = 2$, all the correlation functions have been evaluated as a function of the degree of inelasticity, ϵ , finding a good agreement with Molecular Dynamics simulation results. Also, the two-time correlations have been evaluated.

At this point it is convenient to analyze the main analogies and differences between the HCS and the USF state. In both cases, there is not a Fluctuation-Dissipation relation of the second kind, as the expression for the auto-correlation function of the total internal energy

is not directly related with the coefficients of the macroscopic equation, that in this case is the cooling rate [18]. Moreover, in both the HCS state and the USF state, the two-time correlation function does decay as a homogeneous macroscopic perturbation, see Eq. (50), so that there is a Fluctuation-Dissipation relation of the first kind. The main difference between the two cases resides in the nature of approximation given in Eq. (111). While in the HCS case, the approximate eigenfunction can be identified looking at the linearized homogeneous hydrodynamic equations, in the USF case the equations for the pressure tensor components are needed. Although, in principle, this fact does not have direct consequences at the level of macroscopic hydrodynamics, it is important at the level of the fluctuations. Actually, the correlation function, $\langle \delta \mathcal{P}_{xy}(t) \delta \mathcal{P}_{xy}(0) \rangle$, does not decay as a pure kinetic mode as is the case in the HCS and as was assumed in [18] and, then, the fluctuating quantity $\delta \mathcal{P}_{xy}(t)$ can not be treated simply as a noise in a consistent way (one of the conditions for the results of [18] to hold was that the correlation function of the noise decay faster than the one of the energy). Let us stress that, as the hydrodynamic part of the correlation function is of ϵ^2 order, the coupling disappears in the elastic limit where we exactly recover the result of [18].

Finally, let us mention that many of the general properties shown in the paper can appear in any system beyond Navier-Stokes. Moreover, these results present the starting point for the complete study of the hydrodynamic fluctuating fields in the USF state.

VIII. ACKNOWLEDGMENTS

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Appendix A: Evolution equation for the temperature

The objective of this appendix is to identify the mode that emerges after a homogeneous perturbation of the density and the temperature. Assuming that the hydrodynamic stage has been reached and the distribution function is the one of the time-dependent USF state, we have

$$\frac{dT_H(t)}{dt} = -\frac{2a}{dn_H} P_{xy,H}(t) - \zeta_H(t) T_H(t) \quad (\text{A1})$$

where the pressure tensor and cooling rate can be written as

$$P_{xy,H}(t) = \frac{1}{2}n_H m v_H(t)^2 \tilde{P}_{xy}(\tilde{a}), \quad (\text{A2})$$

$$\zeta_H(t) = \frac{v_H(t)}{\lambda} \tilde{\zeta}(\tilde{a}), \quad (\text{A3})$$

where $\tilde{P}_{xy}(\tilde{a})$ and $\tilde{\zeta}(\tilde{a})$ are defined in Eqs. (68) and (69) respectively.

The deviation

$$\delta \left[\frac{P_{xy,H}}{n_H} \right] \equiv \frac{P_{xy,H}}{n_H} - \frac{P_{xy,s}}{n_s}, \quad (\text{A4})$$

around the stationary value given by n_s and T_s is, to linear order,

$$\delta \left[\frac{P_{xy,H}}{n_H} \right] = \frac{1}{2} m v_s^2 \tilde{P}_{xy,s} \theta - \frac{1}{2} m v_s^2 \tilde{a}_s \frac{\partial \tilde{P}_{xy}}{\partial \tilde{a}}(\tilde{a}_s) \left(\frac{1}{2} \theta + \rho \right), \quad (\text{A5})$$

where we have used that

$$\left[\frac{\partial \tilde{a}}{\partial v_H} \right]_{v_H=v_s} = -\frac{\tilde{a}_s}{v_s}, \quad \left[\frac{\partial \tilde{a}}{\partial n_H} \right]_{v_H=v_s} = -\frac{\tilde{a}_s}{n_s}. \quad (\text{A6})$$

Analogously, for the cooling rate term we have

$$\delta[\zeta_H(t)T_H(t)] = T_s \frac{v_s}{\lambda} \left[\frac{3}{2} \tilde{\zeta}(\tilde{a}_s) - \frac{\tilde{a}_s}{2} \frac{d\tilde{\zeta}}{d\tilde{a}}(\tilde{a}_s) \right] \theta + T_s \frac{v_s}{\lambda} \left[\tilde{\zeta}(\tilde{a}_s) - \tilde{a}_s \frac{d\tilde{\zeta}}{d\tilde{a}}(\tilde{a}_s) \right] \rho. \quad (\text{A7})$$

Taking into account Eqs. (A5) and (A7), we obtain

$$\begin{aligned} \frac{d\theta}{ds} &= \left[2 \frac{\tilde{a}_s^2}{d} \frac{\partial \tilde{P}_{xy}}{\partial \tilde{a}}(\tilde{a}_s) - \tilde{\zeta}_s + \tilde{a}_s \frac{d\tilde{\zeta}}{d\tilde{a}}(\tilde{a}_s) \right] \rho \\ &+ \left[\frac{\tilde{a}_s^2}{d} \frac{\partial \tilde{P}_{xy}}{\partial \tilde{a}}(\tilde{a}_s) - \frac{2\tilde{a}_s}{d} \tilde{P}_{xy,s} - \frac{3}{2} \tilde{\zeta}_s + \frac{1}{2} \tilde{a}_s \frac{d\tilde{\zeta}}{d\tilde{a}}(\tilde{a}_s) \right] \theta. \end{aligned} \quad (\text{A8})$$

But, as in the stationary state we have

$$\frac{2\tilde{a}_s}{d} \tilde{P}_{xy,s} = -\tilde{\zeta}_s, \quad (\text{A9})$$

we obtain the result of the main text, Eq. (66).

Appendix B: Consistency of the approximation given by Eq. (111)

In this Appendix we prove that the approximation given by Eq. (111) is consistent with the equation for the correlation function, Eq. (49). Taking the scalar product with $\bar{\xi}_j(\mathbf{c}_1)$ in Eq. (49) and performing the approximation (111), it is obtained

$$\lambda_j \langle \bar{\xi}_j(\mathbf{c}_1) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle = \langle \bar{\xi}_j(\mathbf{c}_1) | \tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2) \chi(\mathbf{c}_1) \chi(\mathbf{c}_2) \rangle, \quad (\text{B1})$$

whose consistency can be proved. Here we will use a different approach. Eq. (B1) can be obtained by first integrating with respect to \mathbf{c}_2 in Eq. (49) and then take the scalar product with $\bar{\xi}_j(\mathbf{c}_1)$ in the \mathbf{c}_1 space. Once the first step is done, it is obtained

$$\Lambda(\mathbf{c})\chi(\mathbf{c}) = -\tilde{a}_s c_y \frac{\partial}{\partial c_x} \chi(\mathbf{c}), \quad (\text{B2})$$

where we have used that

$$\int d\mathbf{c}_2 \phi(\mathbf{c}_1, \mathbf{c}_2) = -\chi(\mathbf{c}). \quad (\text{B3})$$

Note that Eq. (B2) is nothing but the non-linear Boltzmann equation for the stationary state, but expressed in terms of the linearized Boltzmann operator. And then, approximation (111) has to be consistent when applied to Eq. (B2), so that what we have to prove is

$$\lambda_i \langle \bar{\xi}_i(\mathbf{c}) | \chi(\mathbf{c}) \rangle = \tilde{a}_s \langle \bar{\xi}_i(\mathbf{c}) | c_y \frac{\partial}{\partial c_x} \chi(\mathbf{c}) \rangle, \quad i = 2, 3, 4. \quad (\text{B4})$$

For $i = 1$ the previous equation trivially holds.

Eqs. (B4) can be expressed in a different basis of the subspace $\{\bar{\xi}_i\}_{i=1}^4$. It turns useful to use the basis $\{h_i\}_{i=1}^4$, where

$$\mathbf{h}(\mathbf{c}) = \begin{bmatrix} 1 \\ \frac{2c^2}{d} - 1 \\ 2c_x c_y \\ 2c_y^2 \end{bmatrix}, \quad (\text{B5})$$

because we have

$$\int d\mathbf{c} \mathbf{h}(\mathbf{c}) \delta\chi(\mathbf{c}, s) = \begin{bmatrix} \rho \\ \theta(s) \\ \Pi_{xy}(s) \\ \Pi_{yy}(s) \end{bmatrix}, \quad (\text{B6})$$

and then, Eqs. (B4) can be written as

$$M \begin{bmatrix} 1 \\ 0 \\ \tilde{P}_{xy,s} \\ \tilde{P}_{yy,s} \end{bmatrix} = -\tilde{a}_s \begin{bmatrix} 0 \\ \frac{2}{d} \tilde{P}_{xy,s} \\ \tilde{P}_{yy,s} \\ 0 \end{bmatrix}, \quad (\text{B7})$$

where we have used that

$$\int d\mathbf{c} \chi(\mathbf{c}) = \begin{bmatrix} 1 \\ 0 \\ \tilde{P}_{xy,s} \\ \tilde{P}_{yy,s} \end{bmatrix}, \quad \int d\mathbf{c} \chi(\mathbf{c}) c_y \frac{\partial}{\partial c_x} \chi(\mathbf{c}) = - \begin{bmatrix} 0 \\ \frac{2}{d} \tilde{P}_{xy,s} \\ \tilde{P}_{yy,s} \\ 0 \end{bmatrix}. \quad (\text{B8})$$

Taking the explicit expressions of M , $\tilde{P}_{xy,s}$, and $\tilde{P}_{yy,s}$ as a function of $\tilde{\zeta}_s$ and β of the main text, it is straightforward to prove the validity of Eq. (B7).

Appendix C: Evaluation of the C_{ij} coefficients

In this Appendix we calculate the coefficients C_{ij} defined in Eq. (122), starting from Eq. (124). As said, we only have 6 independent equations, because of the relation between the third and fourth mode. The corresponding equation to $i = j = 2$ is

$$2\lambda_2 \langle \bar{\xi}_2(\mathbf{c}_1) \bar{\xi}_2(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle = \langle \bar{\xi}_2(\mathbf{c}_1) \bar{\xi}_2(\mathbf{c}_2) | \tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2) \chi(\mathbf{c}_1) \chi(\mathbf{c}_2) \rangle. \quad (\text{C1})$$

For $i = 2$ and $j = 3$ we have two equations, one associated to the real part

$$\begin{aligned} (\lambda_2 + \lambda_3^R) \langle \bar{\xi}_2(\mathbf{c}_1) \bar{\xi}_3^R(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle - \lambda_3^I \langle \bar{\xi}_2(\mathbf{c}_1) \bar{\xi}_3^I(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle \\ = \langle \bar{\xi}_2(\mathbf{c}_1) \bar{\xi}_3^R(\mathbf{c}_2) | \tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2) \chi(\mathbf{c}_1) \chi(\mathbf{c}_2) \rangle, \end{aligned} \quad (\text{C2})$$

and other to the imaginary part

$$\begin{aligned} (\lambda_2 + \lambda_3^R) \langle \bar{\xi}_2(\mathbf{c}_1) \bar{\xi}_3^I(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle - \lambda_3^I \langle \bar{\xi}_2(\mathbf{c}_1) \bar{\xi}_3^R(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle \\ = \langle \bar{\xi}_2(\mathbf{c}_1) \bar{\xi}_3^I(\mathbf{c}_2) | \tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2) \chi(\mathbf{c}_1) \chi(\mathbf{c}_2) \rangle, \end{aligned} \quad (\text{C3})$$

where we have used the decomposition into the real and imaginary part of the third eigenvalue and eigenfunctions given by Eqs. (109)-(110). For $i = j = 3$ we also have two independent equations

$$\begin{aligned} 2\lambda_3^R [\langle \bar{\xi}_3^R(\mathbf{c}_1) \bar{\xi}_3^R(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle - \langle \bar{\xi}_3^I(\mathbf{c}_1) \bar{\xi}_3^I(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle] \\ - 4\lambda_3^I \langle \bar{\xi}_3^R(\mathbf{c}_1) \bar{\xi}_3^I(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle \\ = \langle \bar{\xi}_3^R(\mathbf{c}_1) \bar{\xi}_3^R(\mathbf{c}_2) | \tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2) \chi(\mathbf{c}_1) \chi(\mathbf{c}_2) \rangle - \langle \bar{\xi}_3^I(\mathbf{c}_1) \bar{\xi}_3^I(\mathbf{c}_2) | \tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2) \chi(\mathbf{c}_1) \chi(\mathbf{c}_2) \rangle, \end{aligned} \quad (\text{C4})$$

and

$$\begin{aligned} & \lambda_3^I [\langle \bar{\xi}_3^R(\mathbf{c}_1) \bar{\xi}_3^R(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle - \langle \bar{\xi}_3^I(\mathbf{c}_1) \bar{\xi}_3^I(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle] \\ & + 2\lambda_3^R \langle \bar{\xi}_3^R(\mathbf{c}_1) \bar{\xi}_3^I(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle = \langle \bar{\xi}_3^R(\mathbf{c}_1) \bar{\xi}_3^I(\mathbf{c}_2) | \tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2) \chi(\mathbf{c}_1) \chi(\mathbf{c}_2) \rangle. \end{aligned} \quad (\text{C5})$$

Finally, there is an additional equation corresponding to $i = 3, j = 4$

$$\begin{aligned} & 2\lambda_3^R \langle \bar{\xi}_3^R(\mathbf{c}_1) \bar{\xi}_3^R(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle + \langle \bar{\xi}_3^I(\mathbf{c}_1) \bar{\xi}_3^I(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle \\ & = \langle \bar{\xi}_3^R(\mathbf{c}_1) \bar{\xi}_3^R(\mathbf{c}_2) | \tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2) \chi(\mathbf{c}_1) \chi(\mathbf{c}_2) \rangle + \langle \bar{\xi}_3^I(\mathbf{c}_1) \bar{\xi}_3^I(\mathbf{c}_2) | \tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2) \chi(\mathbf{c}_1) \chi(\mathbf{c}_2) \rangle, \end{aligned} \quad (\text{C6})$$

that can be written in terms of the third mode because $\lambda_4 = \lambda_3^*$ and $\bar{\xi}_4 = \bar{\xi}_3^*$.

The scalar products $\langle \bar{\xi}_i(\mathbf{c}_1) \bar{\xi}_j(\mathbf{c}_2) | \phi(\mathbf{c}_1, \mathbf{c}_2) \rangle$ can be written in terms of the C_{ij} coefficients through Eq. (125), so that the system of equations (C1)-(C6) is a linear system of six equations for the six unknown coefficients $\{C_{22}, C_{23}, C_{24}, C_{33}, C_{34}, C_{44}\}$. Let us note that, until now, the results are valid for any dimension, d , and the only approximation made was the one given by Eq. (111). Of course, it still remains to evaluate the coefficients

$$\langle \bar{\xi}_i(\mathbf{c}_1) \bar{\xi}_j(\mathbf{c}_2) | \tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2) \chi(\mathbf{c}_1) \chi(\mathbf{c}_2) \rangle = \sum_{l=1}^4 \xi_{il} \xi_{jl} T_{ll} + \sum_{k>l=1}^4 (\xi_{ik} \xi_{jl} + \xi_{il} \xi_{jk}) T_{kl}, \quad (\text{C7})$$

where we have introduced the matrix elements

$$T_{ij} = \int d\mathbf{c}_1 \int d\mathbf{c}_2 b_i(\mathbf{c}_1) b_j(\mathbf{c}_2) \tilde{T}_0(\mathbf{c}_1, \mathbf{c}_2) \chi(\mathbf{c}_1) \chi(\mathbf{c}_2) = \int d\mathbf{c}_1 \int d\mathbf{c}_2 \chi(\mathbf{c}_1) \chi(\mathbf{c}_2) T_0(\mathbf{c}_1, \mathbf{c}_2) b_i(\mathbf{c}_1) b_j(\mathbf{c}_2), \quad (\text{C8})$$

with

$$T_0(\mathbf{c}_1, \mathbf{c}_2) = \int d\hat{\boldsymbol{\sigma}} \Theta(\mathbf{c}_{12} \cdot \hat{\boldsymbol{\sigma}}) (\mathbf{c}_{12} \cdot \hat{\boldsymbol{\sigma}}) [b_{\boldsymbol{\sigma}}(1, 2) - 1]. \quad (\text{C9})$$

The first coefficients, $\{T_{1j}\}_{j=1}^4$, can be easily calculated. In effect,

$$T_{11} = 0, \quad (\text{C10})$$

due to the conservation of the total number of particles and the second is related with the cooling rate

$$T_{12} = -\frac{d}{2} \tilde{\zeta}_s, \quad (\text{C11})$$

by Eq. (69). On the other hand, taking into account the equation for χ , Eq. (B2), we have

$$T_{13} = -\tilde{a}_s \int d\mathbf{c} c_x c_y^2 \frac{\partial}{\partial c_x} \chi(\mathbf{c}) = \frac{1}{2} \tilde{a}_s \tilde{P}_{yy,s}, \quad (\text{C12})$$

and

$$T_{14} = -\tilde{a}_s \int d\mathbf{c} c_y^2 \frac{\partial}{\partial c_x} \chi(\mathbf{c}) = 0. \quad (\text{C13})$$

To evaluate the other coefficients we have to calculate explicitly $T_0(\mathbf{c}_1, \mathbf{c}_2) b_i(\mathbf{c}_1) b_j(\mathbf{c}_2)$. In fact, in reference [9] the term $T_0(\mathbf{c}_1, \mathbf{c}_2) c_1^2 c_2^2$ was already calculated obtaining

$$T_0(\mathbf{c}_1, \mathbf{c}_2) c_1^2 c_2^2 = -\frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d+5}{2}\right)} \left[\frac{(1-\alpha^2)(d+1+2\alpha^2)}{16} g^5 + \frac{d+5-\alpha^2(d+1)+4\alpha}{4} g^3 G^2 - \frac{1+\alpha}{2} (2d+3-3\alpha) g(\mathbf{g} \cdot \mathbf{G})^2 \right], \quad (\text{C14})$$

where we have introduced the new variables

$$\mathbf{g} = \mathbf{c}_1 - \mathbf{c}_2, \quad (\text{C15})$$

$$\mathbf{G} = \frac{1}{2}(\mathbf{c}_1 + \mathbf{c}_2). \quad (\text{C16})$$

For the rest of coefficients, we first evaluate $[b_\sigma(1, 2) - 1] b_i(\mathbf{c}_1) b_j(\mathbf{c}_2)$. Using the collision rule, Eq. (1), it is obtained

$$\begin{aligned} [b_\sigma(1, 2) - 1] c_1^2 c_{2x} c_{2y} &= \frac{1+\alpha}{2} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}) [c_1^2 (c_{2x} \hat{\sigma}_y + c_{2y} \hat{\sigma}_x) - 2(\hat{\boldsymbol{\sigma}} \cdot \mathbf{c}_1) c_{2x} c_{2y}] \\ &\quad + \frac{(1+\alpha)^2}{4} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g})^2 [c_1^2 \hat{\sigma}_x \hat{\sigma}_y + c_{2x} c_{2y} - 2(\hat{\boldsymbol{\sigma}} \cdot \mathbf{c}_1) (c_{2x} \hat{\sigma}_y + c_{2y} \hat{\sigma}_x)] \\ &\quad + \frac{(1+\alpha)^3}{8} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g})^3 [c_{2x} \hat{\sigma}_y + c_{2y} \hat{\sigma}_x - 2(\hat{\boldsymbol{\sigma}} \cdot \mathbf{c}_1) \hat{\sigma}_x \hat{\sigma}_y] + \frac{(1+\alpha)^4}{16} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g})^4 \hat{\sigma}_x \hat{\sigma}_y, \end{aligned} \quad (\text{C17})$$

$$\begin{aligned} [b_\sigma(1, 2) - 1] c_1^2 c_{2y}^2 &= (1+\alpha) (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}) [c_1^2 c_{2y} \hat{\sigma}_y - c_{2y}^2 (\hat{\boldsymbol{\sigma}} \cdot \mathbf{c}_1)] \\ &\quad + \frac{(1+\alpha)^2}{4} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g})^2 [c_1^2 \hat{\sigma}_y^2 + c_{2y}^2 - 4(\hat{\boldsymbol{\sigma}} \cdot \mathbf{c}_1) c_{2y} \hat{\sigma}_y] \\ &\quad + \frac{(1+\alpha)^3}{4} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g})^3 [c_{2y} \hat{\sigma}_y - (\hat{\boldsymbol{\sigma}} \cdot \mathbf{c}_1) \hat{\sigma}_y^2] + \frac{(1+\alpha)^4}{16} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g})^4 \hat{\sigma}_y^2, \end{aligned} \quad (\text{C18})$$

$$\begin{aligned} [b_\sigma(1, 2) - 1] c_{1x} c_{1y} c_{2x} c_{2y} &= \frac{1+\alpha}{2} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}) (\hat{\sigma}_y c_{1x} c_{2x} g_y + \hat{\sigma}_x c_{1y} c_{2y} g_x) \\ &\quad + \frac{(1+\alpha)^2}{4} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g})^2 [\hat{\sigma}_x \hat{\sigma}_y g_x g_y - \hat{\sigma}_x^2 c_{1y} c_{2y} - \hat{\sigma}_y^2 c_{1x} c_{2x}] \\ &\quad - \frac{(1+\alpha)^3}{8} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g})^3 (\hat{\sigma}_x \hat{\sigma}_y^2 g_x + \hat{\sigma}_x^2 \hat{\sigma}_y g_y) - \frac{(1+\alpha)^4}{16} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g})^4 \hat{\sigma}_x^2 \hat{\sigma}_y^2, \end{aligned} \quad (\text{C19})$$

$$\begin{aligned} [b_\sigma(1, 2) - 1] c_{1x} c_{1y} c_{2y}^2 &= \frac{1+\alpha}{2} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g}) [\hat{\sigma}_y (2c_{1x} c_{1y} c_{2y} - c_{1x} c_{2y}^2) - \hat{\sigma}_x c_{1y} c_{2y}^2] \\ &\quad + \frac{(1+\alpha)^2}{4} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g})^2 [\hat{\sigma}_y^2 (c_{1x} c_{1y} - 2c_{1x} c_{2y}) + \hat{\sigma}_x \hat{\sigma}_y (c_{2y}^2 - 2c_{1y} c_{2y})] \\ &\quad + \frac{(1+\alpha)^3}{8} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g})^3 [\hat{\sigma}_x \hat{\sigma}_y^2 (2c_{2y} - c_{1y}) - \hat{\sigma}_y^3 c_{1x}] + \frac{(1+\alpha)^4}{16} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{g})^4 \hat{\sigma}_x \hat{\sigma}_y^3, \end{aligned} \quad (\text{C20})$$

and

$$\begin{aligned}
[b_{\sigma}(1, 2) - 1]c_{1y}^2c_{2y}^2 &= (1 + \alpha)(\hat{\sigma} \cdot \mathbf{g})\hat{\sigma}_y(c_{1y}^2c_{2y} - c_{1y}c_{2y}^2) \\
&\quad + \frac{(1 + \alpha)^2}{4}(\hat{\sigma} \cdot \mathbf{g})^2\hat{\sigma}_y^2(c_{1y}^2 + c_{2y}^2 - 4c_{1y}c_{2y}) \\
&\quad - \frac{(1 + \alpha)^3}{4}(\hat{\sigma} \cdot \mathbf{g})^3\hat{\sigma}_y^3g_y + \frac{(1 + \alpha)^4}{16}(\hat{\sigma} \cdot \mathbf{g})^4\hat{\sigma}_y^4.
\end{aligned} \tag{C21}$$

After multiplying by $\hat{\sigma} \cdot \mathbf{g}$, the $\hat{\sigma}$ -integrals can be calculated with the aid of

$$\int d\hat{\sigma}\Theta(\hat{\sigma} \cdot \mathbf{g})(\hat{\sigma} \cdot \mathbf{g})^2\hat{\sigma}_i = \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+3}{2})}gg_i, \tag{C22}$$

$$\int d\hat{\sigma}\Theta(\hat{\sigma} \cdot \mathbf{g})(\hat{\sigma} \cdot \mathbf{g})^3 = \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+3}{2})}g^3, \tag{C23}$$

$$\int d\hat{\sigma}\Theta(\hat{\sigma} \cdot \mathbf{g})(\hat{\sigma} \cdot \mathbf{g})^3\hat{\sigma}_i\hat{\sigma}_j = \frac{\pi^{\frac{d-1}{2}}}{2\Gamma(\frac{d+5}{2})}(3gg_ig_j + g^3\delta_{ij}), \tag{C24}$$

$$\int d\hat{\sigma}\Theta(\hat{\sigma} \cdot \mathbf{g})(\hat{\sigma} \cdot \mathbf{g})^4\hat{\sigma}_i = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+5}{2})}g^3g_i, \tag{C25}$$

$$\int d\hat{\sigma}\Theta(\hat{\sigma} \cdot \mathbf{g})(\hat{\sigma} \cdot \mathbf{g})^4\hat{\sigma}_y^2\hat{\sigma}_j = \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+7}{2})}[3gg_y^2g_j + g^3g_j + 2g^3g_y\delta_{yj}], \tag{C26}$$

$$\int d\hat{\sigma}\Theta(\hat{\sigma} \cdot \mathbf{g})(\hat{\sigma} \cdot \mathbf{g})^4\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z = \frac{3\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+7}{2})}gg_xg_yg_z, \tag{C27}$$

$$\int d\hat{\sigma}\Theta(\hat{\sigma} \cdot \mathbf{g})(\hat{\sigma} \cdot \mathbf{g})^5\hat{\sigma}_i\hat{\sigma}_j = \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+7}{2})}(5g^3g_ig_j + g^5\delta_{ij}), \tag{C28}$$

$$\int d\hat{\sigma}\Theta(\hat{\sigma} \cdot \mathbf{g})(\hat{\sigma} \cdot \mathbf{g})^5\hat{\sigma}_y^3\hat{\sigma}_j = \frac{3\pi^{\frac{d-1}{2}}}{2\Gamma(\frac{d+9}{2})}[5gg_y(g^2 + g_y^2)g_j + g^3(g^2 + 5g_y^2)\delta_{yj}], \tag{C29}$$

calculated for arbitrary dimension and

$$\int d\hat{\sigma}\Theta(\hat{\sigma} \cdot \mathbf{g})(\hat{\sigma} \cdot \mathbf{g})^5\hat{\sigma}_x^2\hat{\sigma}_j^2 = \frac{3\pi^{\frac{1}{2}}}{2\Gamma(\frac{11}{2})}[g^3(g^2 + g_x^2) + 8g^3g_xg_j\delta_{xj} + (g^3 + 5gg_x^2)g_j^2], \tag{C30}$$

for $d = 2$.

In the following we will restrict ourselves to the case $d = 2$ and will use the Jenkins and Richman distribution to order ϵ^2 [20]

$$\chi(\mathbf{c}) \approx \frac{e^{-c^2}}{\pi} \left[1 - \epsilon\sqrt{2}c_xc_y + \epsilon^2 \left(\frac{1}{4} - c_y^2 + c_x^2c_y^2 \right) \right]. \tag{C31}$$

To the same order, we have

$$\begin{aligned} \chi(\mathbf{c}_1)\chi(\mathbf{c}_2) &\approx \frac{1}{\pi^2} e^{-c_1^2 - c_2^2} \left[1 - \epsilon\sqrt{2}(c_{1x}c_{1y} + c_{2x}c_{2y}) \right. \\ &\quad \left. + \epsilon^2 \left(\frac{1}{4} - c_{1y}^2 - c_{2y}^2 + c_{1x}^2 c_{1y}^2 + c_{2x}^2 c_{2y}^2 + 2c_{1x}c_{1y}c_{2x}c_{2y} \right) \right], \end{aligned} \quad (\text{C32})$$

or, in terms of the new variables $\{\mathbf{g}, \mathbf{G}\}$

$$\begin{aligned} \chi(\mathbf{c}_1)\chi(\mathbf{c}_2) &\approx \frac{e^{-\frac{1}{2}g^2 - 2G^2}}{\pi^2} \left\{ 1 - \frac{\epsilon}{\sqrt{2}}(g_x g_y + 4G_x G_y) \right. \\ &\quad \left. + \frac{\epsilon^2}{4} [2 + (g_x^2 - 2)g_y^2 + 8g_x g_y G_x G_y + 8(2G_x^2 - 1)G_y^2] \right\}. \end{aligned} \quad (\text{C33})$$

The velocity integrals given by Eq. (C8) can be calculated with the aid of Mathematica, obtaining to ϵ^2 order

$$T_{22} = -\frac{3}{2}\sqrt{\frac{\pi}{2}}\epsilon^2, \quad (\text{C34})$$

$$T_{23} = \frac{5}{8}\sqrt{\pi}\epsilon, \quad (\text{C35})$$

$$T_{24} = -\frac{1}{8}\sqrt{\frac{\pi}{2}}\epsilon^2, \quad (\text{C36})$$

$$T_{33} = -\frac{19}{64}\sqrt{\frac{\pi}{2}}\epsilon^2, \quad (\text{C37})$$

$$T_{34} = \frac{5}{16}\sqrt{\pi}\epsilon, \quad (\text{C38})$$

$$T_{44} = \frac{11}{64}\sqrt{\frac{\pi}{2}}\epsilon^2. \quad (\text{C39})$$

Finally, by substituting the obtained expressions of the T_{ij} coefficients into Eq. (C7) and that into Eqs. (C1)-(C6), we obtain the above mentioned linear system for C_{ij} . This system is solved with the aid of Mathematica obtaining the expressions of the main text.

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